Network Algorithms
(Lecture Notes)

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Introduction

Welcome to the Network Algorithms course!

Computer systems are becoming more distributed and networked. This has two main implications: First, it becomes more difficult to overview the system and reason about its correctness. Second, in order to fully exploit the distributed resource network, efficient algorithms need to be designed.

In this course, you will learn techniques to design and analyze communication-efficient algorithms for networks. We will start with most simple tasks, such as electing a leader or computing a spanning tree in a network, and then move to more complex tasks such as network coloring, synchronization, or medium access.

We will see that many classic problems from Computer Science have their analoga and applications in the distributed world. Often, these problems are solved in the network to make a more efficient use of the network resources: for example, spanning trees form a helpful communication infrastructure and are used to disseminate and aggregate information efficiently, graph colorings are used to parallelize computations and reuse e.g. the wireless frequency spectrum, matchings are used for load-balancing purposes, dominating sets are used to construct backbones in ad-hoc networks, etc. However, classic CS problems are also solved in the network to speed up computations: for example, while on a single machine sorting a set of \( n \) values takes time \( \Theta(n \log n) \), the set can be sorted in time \( O(\log n) \) in a network.

You will study fundamental metrics to evaluate network algorithms, such as the (distributed) runtime or the communication complexity, and you will learn how to analyze them formally. Moreover, you will also learn a simple synchronous model to develop and reason about efficient network algorithms: once formulated in this model, your algorithm can be automatically transformed into a robust algorithm which is correct in asynchronous and even adversarial environments.

The goal of this course is to cover a different topic in each lecture, to highlight common themes and techniques to design both deterministic as well as randomized network algorithms, and also to give insights into the what is not possible in a network (lower bounds and impossibility results).

Have fun!                      Stefan Schmid (Berlin, October 2013)
Chapter 1

Communication Networks

The most basic network topologies used in practice are trees, rings, grids or tori. Many other suggested networks are simply combinations or derivatives of these. The advantage of trees is that the routing is very easy: for every source-destination pair there is only one possible simple path. However, since the root of a tree is usually a severe bottleneck, so-called fat trees have been used. These trees have the property that every edge connecting a node $v$ to its parent $u$ has a capacity that is equal to all leaves of the subtree routed at $v$. See Figure 1.1 for an example.

![Figure 1.1: The structure of a fat tree.](image)

Remarks:

- Fat trees belong to a family of networks that require edges of non-uniform capacity to be efficient. Easier to build are networks with edges of uniform capacity. This is usually the case for grids and tori. Unless explicitly mentioned, we will henceforth treat all edges to be of capacity 1. In the following, $[x]$ means the set $\{0, \ldots, x-1\}$.

**Definition 1.1** (Torus, Mesh). Let $m, d \in \mathbb{N}$. The $(m, d)$-mesh $M(m, d)$ is a graph
with node set $V = [m]^d$ and edge set

$$E = \left\{ ((a_1, \ldots, a_d), (b_1, \ldots, b_d)) \mid a_i, b_i \in [m], \sum_{i=1}^{d} |a_i - b_i| = 1 \right\}.$$

The $(m, d)$-torus $T(m, d)$ is a graph that consists of an $(m, d)$-mesh and additionally wrap-around edges from nodes $(a_1, \ldots, a_{i-1}, m, a_{i+1}, \ldots, a_d)$ to nodes $(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_d)$ for all $i \in \{1, \ldots, d\}$ and all $a_j \in [m]$ with $j \neq i$. In other words, we take the expression $a_i - b_i$ in the sum modulo $m$ prior to computing the absolute value. $M(m, 1)$ is also called a line, $T(m, 1)$ a cycle, and $M(2, d) = T(2, d)$ a $d$-dimensional hypercube. Figure 1.2 presents a linear array, a torus, and a hypercube.

![Figure 1.2: The structure of $M(m, 1)$, $T(4, 2)$, and $M(2, 3)$](image)

**Remarks:**

- Routing on mesh, torus, and hypercube is trivial. On a $d$-dimensional hypercube, to get from a source bitstring $s$ to a target bitstring $d$ one only needs to fix each “wrong” bit, one at a time; in other words, if the source and the target differ by $k$ bits, there are $k!$ routes with $k$ hops.

- The hypercube can be used for a structured P2P architecture (a distributed hash table (DHT)): We have $n$ nodes, $n$ for simplicity being a power of 2, i.e., $n = 2^d$. As in the hypercube, each node gets a unique $d$-bit ID, and each node connects to $d$ other nodes, i.e., the nodes that have IDs differing in exactly one bit. Now we use a globally known hash function $f$, mapping file names to long bit strings; SHA-1 is popular in practice, providing 160 bits. Let $f_d$ denote the first $d$ bits (prefix) of the bitstring produced by $f$. If a node is searching for file name $X$, it routes a request message $f(X)$ to node $f_d(X)$. Clearly, node $f_d(X)$ can only answer this request if all files with hash prefix $f_d(X)$ have been previously registered at node $f_d(X)$.

- There are a few issues which need to be addressed before our DHT works, in particular churn (nodes joining and leaving without notice). To deal with churn the system needs some level of replication, i.e., a number of nodes which are responsible for each prefix such that failure of some nodes will not compromise the system. In addition there are other issues (e.g., security, efficiency) which
can be addressed to improve the system. Delay efficiency for instance is already considered in the seminal paper by Plaxton et al. These issues are beyond the scope of this lecture.

- The hypercube has many derivatives, the so-called hypercubic networks. Among these are the butterfly, cube-connected-cycles, shuffle-exchange, and de Bruijn graph. We start with the butterfly, which is basically a “rolled out” hypercube (hence directly providing replication!).

**Definition 1.2 (Butterfly).** Let $d \in N$. The $d$-dimensional butterfly $BF(d)$ is a graph with node set $V = [d + 1] \times [2]^d$ and an edge set $E = E_1 \cup E_2$ with

$$E_1 = \{(i, \alpha), (i + 1, \alpha) \mid i \in [d], \alpha \in [2]^d\}$$

and

$$E_2 = \{(i, \alpha), (i + 1, \beta) \mid i \in [d], \alpha, \beta \in [2]^d, \alpha \text{ and } \beta \text{ differ only at the } i^{th} \text{ position}\}.$$

A node set $\{(i, \alpha) \mid \alpha \in [2]^d\}$ is said to form level $i$ of the butterfly. The $d$-dimensional wrap-around butterfly $W-BF(d)$ is defined by taking the $BF(d)$ and identifying level $d$ with level 0.

**Remarks:**

- Figure 1.3 shows the 3-dimensional butterfly $BF(3)$. The $BF(d)$ has $(d + 1)2^d$ nodes, $2d \cdot 2^d$ edges and degree 4. It is not difficult to check that combining the node sets $\{(i, \alpha) \mid i \in [d]\}$ into a single node results in the hypercube.

- Butterflies have the advantage of a constant node degree over hypercubes, whereas hypercubes feature more fault-tolerant routing.

- Although butterflies are used in the P2P context (e.g. Viceroy), they have been used decades earlier for communication switches. The well-known Benes network is nothing but two back-to-back butterflies. And indeed, butterflies (and other hypercubic networks) are even older than that; students familiar with fast fourier transform (FFT) may recognize the structure. Every year there is a new application for which a hypercubic network is the perfect solution!

- Indeed, hypercubic networks are related. Since all structured P2P architectures are based on hypercubic networks, they in turn are all related.

- Next we define the cube-connected-cycles network. It only has a degree of 3 and it results from the hypercube by replacing the corners by cycles.

**Definition 1.3 (Cube-Connected-Cycles).** Let $d \in N$. The cube-connected-cycles network $CCC(d)$ is a graph with node set $V = \{(a, p) \mid a \in [2]^d, p \in [d]\}$ and edge set

$$E = \{(a, p), (a, (p + 1) \mod d) \mid a \in [2]^d, p \in [d]\} \cup \{(a, p), (b, p) \mid a, b \in [2]^d, p \in [d], a = b \text{ except for } a_p\}.$$
CHAPTER 1. COMMUNICATION NETWORKS

Figure 1.3: The structure of BF(3).

Figure 1.4: The structure of CCC(3).

Remarks:

- Two possible representations of a CCC can be found in Figure 1.4.
- The shuffle-exchange is yet another way of transforming the hypercubic interconnection structure into a constant degree network.

**Definition 1.4** (Shuffle-Exchange). Let \( d \in \mathbb{N} \). The \( d \)-dimensional shuffle-exchange \( SE(d) \) is defined as an undirected graph with node set \( V = [2]^d \) and an edge set \( E = E_1 \cup E_2 \) with

\[
E_1 = \{(a_1, \ldots, a_d), (\bar{a}_1, \ldots, \bar{a}_d) \mid (a_1, \ldots, a_d) \in [2]^d, \bar{a}_d = 1 - a_d \}
\]

and

\[
E_2 = \{(a_1, \ldots, a_d), (a_d, a_1, \ldots, a_{d-1}) \mid (a_1, \ldots, a_d) \in [2]^d \}.
\]

Figure 1.5 shows the 3- and 4-dimensional shuffle-exchange graph.

**Definition 1.5** (DeBruijn). The \( b \)-ary DeBruijn graph of dimension \( d \) \( DB(b, d) \) is an undirected graph \( G = (V, E) \) with node set \( V = \{v \in [b]^d \} \) and edge set \( E \) that
contains all edges \( \{v, w\} \) with the property that \( w \in \{(x, v_1, \ldots, v_{d-1}) : x \in [b]\} \), where \( v = (v_1, \ldots, v_d) \).

Remarks:

- Two examples of a DeBruijn graph can be found in Figure 1.6. The DeBruijn graph is the basis of the Koorde P2P architecture.

- There are some data structures which also qualify as hypercubic networks. An obvious example is the Chord P2P architecture, which uses a slightly different hypercubic topology. A less obvious (and therefore good) example is the skip list, the balanced binary search tree for the lazy programmer:

**Definition 1.6 (Skip List).** The skip list is an ordinary ordered linked list of objects, augmented with additional forward links. The ordinary linked list is the level 0 of the skip list. In addition, every object is promoted to level 1 with probability \( 1/2 \). As for level 0, all level 1 objects are connected by a linked list. In general, every object on level \( i \) is promoted to the next level with probability \( 1/2 \). A special start-object points to the smallest/first object on each level.

Remarks:

- Search, insert, and delete can be implemented in \( O(\log n) \) expected time in a skip list, simply by jumping from higher levels to lower ones when overshooting the searched position. Also, the amortized memory cost of each object is constant, as on average an object only has two forward pointers.
• The randomization can easily be discarded, by deterministically promoting a constant fraction of objects of level \( i \) to level \( i + 1 \), for all \( i \). When inserting or deleting, object \( o \) simply checks whether its left and right level \( i \) neighbors are being promoted to level \( i + 1 \). If none of them is, promote object \( o \) itself. Essentially we establish a maximal independent set on each level, hence at least every third and at most every second object is promoted.

• There are obvious variants of the skip list, e.g., the skip graph. Instead of promoting only half of the nodes to the next level, we always promote all the nodes, similarly to a balanced binary tree: All nodes are part of the root level of the binary tree. Half the nodes are promoted left, and half the nodes are promoted right, on each level. Hence on level \( i \) we have have \( 2^i \) lists (or, more symmetrically: rings) of about \( n/2^i \) objects. This is pretty much what we need for a nice hypercubic P2P architecture.

• One important goal in choosing a topology for a network is that it has a small diameter. The following theorem presents a lower bound for this.

**Theorem 1.7.** Every graph of maximum degree \( d > 2 \) and size \( n \) must have a diameter of at least \( \lceil \log n / \log(d - 1) \rceil - 2 \).

**Proof.** Suppose we have a graph \( G = (V, E) \) of maximum degree \( d \) and size \( n \). Start from any node \( v \in V \). In a first step at most \( d \) other nodes can be reached. In two steps at most \( d \cdot (d - 1) \) additional nodes can be reached. Thus, in general, in at most \( k \) steps at most

\[
1 + \sum_{i=0}^{k-1} d \cdot (d - 1)^i = 1 + d \cdot \frac{(d - 1)^k - 1}{d - 1} - 1 \leq \frac{d \cdot (d - 1)^k}{d - 2}
\]

nodes (including \( v \)) can be reached. This has to be at least \( n \) to ensure that \( v \) can reach all other nodes in \( V \) within \( k \) steps. Hence,

\[
(d - 1)^k \geq \frac{(d - 2) \cdot n}{d} \quad \Leftrightarrow \quad k \geq \log_{d-1}((d-2) \cdot n/d).
\]

Since \( \log_{d-1}((d-2)/d) > -2 \) for all \( d > 2 \), this is true only if

\[
k \geq \lceil \log n / \log(d - 1) \rceil - 2.
\]

**Remarks:**

• In other words, constant-degree hypercubic networks feature an asymptotically optimal diameter.

• There are a few other interesting graph classes, e.g., expander graphs (an expander graph is a sparse graph which has high connectivity properties, that is, from every not too large subset of nodes you are connected to a larger set of nodes), or small-world graphs (popular representations of social networks). At first sight hypercubic networks seem to be related to expanders and small-world graphs, but they are not.
Chapter 2

Leader Election

2.1 Distributed Algorithms and Complexity

In the second part of this course we will often model the distributed system as a network or graph, and study protocols in which nodes (i.e., the processors) can only communicate with their neighbors to perform certain tasks. We are often interested in the following synchronous model or algorithm.

Definition 2.1 (Synchronous Distributed Algorithm). In a synchronous algorithm, nodes operate in synchronous rounds. In each round, each processor executes the following steps:

1. Do some local computation (of reasonable “local complexity”).
2. Send messages to neighbors in graph (of reasonable size).
3. Receive messages (that were sent by neighbors in step 2 of the same round).

Remarks:

• Any other step ordering is fine.

The other cornerstone model is the asynchronous algorithm.

Definition 2.2 (Asynchronous Distributed Algorithm). In the asynchronous model, algorithms are event driven (“upon receiving message . . . , do . . . ”). Processors cannot access a global clock. A message sent from one processor to another will arrive in finite but unbounded time.

Remarks:

• The asynchronous model and the synchronous model (Definition 2.1) are the cornerstone models in distributed computing. As they do not necessarily reflect reality there are several models in between synchronous and asynchronous. However, from a theoretical point of view the synchronous and the asynchronous model are the most interesting ones (because every other model is in between these extremes).

• Note that in the asynchronous model, messages that take a longer path may arrive earlier.
In order to evaluate an algorithm, apart from the local complexity mentioned above, we consider the following metrics.

**Definition 2.3 (Time Complexity).** For synchronous algorithms (as defined in 2.1) the time complexity is the number of rounds until the algorithm terminates.

**Definition 2.4 (Time Complexity).** For asynchronous algorithms (as defined in 2.1) the time complexity is the number of time units from the start of the execution to its completion in the worst case (every legal input, every execution scenario), assuming that each message has a delay of at most one time unit.

**Remarks:**

- You cannot use the maximum delay in the algorithm design. In other words, the algorithm has to be correct even if there is no such delay upper bound.

**Definition 2.5 (Message Complexity).** The message complexity of a synchronous and asynchronous algorithm is determined by the number of messages exchanged (again every legal input, every execution scenario).

### 2.2 Anonymous Leader Election

Some algorithms (e.g., for medium access) ask for a special node, a so-called “leader”. Computing a leader is a most simple form of symmetry breaking. Algorithms based on leaders do generally not exhibit a high degree of parallelism, and therefore often suffer from poor (parallel) time complexity. However, sometimes it is still useful to have a leader to make critical decisions in an easy (though non-distributed!) way.

The process of choosing a leader is known as *leader election*. Although leader election is a simple form of symmetry breaking, there are some remarkable issues that allow us to introduce notable computational models.

In this chapter we concentrate on the ring topology. The ring is the “drosophila” of distributed computing as many interesting challenges already reveal the root of the problem in the special case of the ring. Paying special attention to the ring also makes sense from a practical point of view as some real world systems are based on a ring topology, e.g., the token ring standard for local area networks.

**Problem 2.6 (Leader Election).** Each node eventually decides whether it is a leader or not, subject to the constraint that there is exactly one leader.

**Remarks:**

- More formally, nodes are in one of three states: undecided, leader, not leader. Initially every node is in the undecided state. When leaving the undecided state, a node goes into a final state (leader or not leader).

**Definition 2.7 (Anonymous).** A system is anonymous if nodes do not have unique identifiers.

**Definition 2.8 (Uniform).** An algorithm is called uniform if the number of nodes \( n \) is not known to the algorithm (to the nodes, if you wish). If \( n \) is known, the algorithm is called non-uniform.
2.3. ASYNCHRONOUS RING

Whether a leader can be elected in an anonymous system depends on whether the network is symmetric (ring, complete graph, complete bipartite graph, etc.) or asymmetric (star, single node with highest degree, etc.). Simplifying slightly, in this context a symmetric graph is a graph in which the extended neighborhood of each node has the same structure. We will now show that non-uniform anonymous leader election for synchronous rings is impossible. The idea is that in a ring, symmetry can always be maintained.

Lemma 2.9. After round $k$ of any deterministic algorithm on an anonymous ring, each node is in the same state $s_k$.

Proof by induction: All nodes start in the same state. A round in a synchronous algorithm consists of the three steps sending, receiving, local computation (see Definition 2.1). All nodes send the same message(s), receive the same message(s), do the same local computation, and therefore end up in the same state.

Theorem 2.10 (Anonymous Leader Election). Deterministic leader election in an anonymous ring is impossible.

Proof (with Lemma 2.9): If one node ever decides to become a leader (or a non-leader), then every other node does so as well, contradicting the problem specification 2.6 for $n > 1$. This holds for non-uniform algorithms, and therefore also for uniform algorithms. Furthermore, it holds for synchronous algorithms, and therefore also for asynchronous algorithms.

Remarks:

• Sense of direction is the ability of nodes to distinguish neighbor nodes in an anonymous setting. In a ring, for example, a node can distinguish the clockwise and the counterclockwise neighbor. Sense of direction does not help in anonymous leader election.

• Theorem 2.10 also holds for other symmetric network topologies (e.g., complete graphs, complete bipartite graphs, ...).

• Note that Theorem 2.10 does not hold for randomized algorithms; if nodes are allowed to toss a coin, symmetries can be broken.

2.3 Asynchronous Ring

We first concentrate on the asynchronous model from Definition 2.2. Throughout this section we assume non-anonymity; each node has a unique identifier as proposed in Assumption 4.2:

Assumption 2.11 (Node Identifiers). Each node has a unique identifier, e.g., its IP address. We usually assume that each identifier consists of only $\log n$ bits if the system has $n$ nodes.

Having ID’s seems to lead to a trivial leader election algorithm, as we can simply elect the node with, e.g., the highest ID.

Theorem 2.12 (Analysis of Algorithm 1). Algorithm 1 is correct. The time complexity is $O(n)$. The message complexity is $O(n^2)$. 
Algorithm 1 Clockwise

1: Each node $v$ executes the following code:
2: $v$ sends a message with its identifier (for simplicity also $v$) to its clockwise neighbor. \{If node $v$ already received a message $w$ with $w > v$, then node $v$ can skip this step; if node $v$ receives its first message $w$ with $w < v$, then node $v$ will immediately send $v$.\}
3: if $v$ receives a message $w$ with $w > v$ then
4: $v$ forwards $w$ to its clockwise neighbor
5: $v$ decides not to be the leader, if it has not done so already.
6: else if $v$ receives its own identifier $v$ then
7: $v$ decides to be the leader
8: end if

Proof: Let node $z$ be the node with the maximum identifier. Node $z$ sends its identifier in clockwise direction, and since no other node can swallow it, eventually a message will arrive at $z$ containing it. Then $z$ declares itself to be the leader. Every other node will declare non-leader at the latest when forwarding message $z$. Since there are $n$ identifiers in the system, each node will at most forward $n$ messages, giving a message complexity of at most $n^2$. We start measuring the time when the first node that “wakes up” sends its identifier. For asynchronous time complexity (Definition 2.4) we assume that each message takes at most one time unit to arrive at its destination. After at most $n - 1$ time units the message therefore arrives at node $z$, waking $z$ up. Routing the message $z$ around the ring takes at most $n$ time units. Therefore node $z$ decides no later than at time $2n - 1$. Every other node decides before node $z$.

Remarks:

- Note that in Algorithm 1 nodes need to distinguish between clockwise and counterclockwise neighbors. In fact they do not: It is okay to simply send your own identifier to any neighbor, and forward a message $m$ to the neighbor you did not receive the message $m$ from. So nodes only need to be able to distinguish their two neighbors.

- Can we improve this algorithm?

Theorem 2.13 (Analysis of Algorithm 2). Algorithm 2 is correct. The time complexity is $O(n)$. The message complexity is $O(n \log n)$.

Proof: Correctness is as in Theorem 2.12. The time complexity is $O(n)$ since the node with maximum identifier $z$ sends messages with round-trip times $2, 4, 8, 16, \ldots, 2 \cdot 2^k$ with $k \leq \log(n + 1)$. (Even if we include the additional wake-up overhead, the time complexity stays linear.) Proving the message complexity is slightly harder: if a node $v$ manages to survive round $r$, no other node in distance $2^r$ (or less) survives round $r$. That is, node $v$ is the only node in its $2^r$-neighborhood that remains active in round $r + 1$. Since this is the same for every node, less than $n/2^r$ nodes are active in round $r + 1$. Being active in round $r$ costs $2 \cdot 2 \cdot 2^r$ messages. Therefore, round $r$ costs at most $2 \cdot 2 \cdot 2^r \cdot \frac{n}{2^r} = 8n$ messages. Since there are only logarithmic many possible rounds, the message complexity follows immediately.
Algorithm 2 Radius Growth (For readability we provide pseudo-code only; for a formal version please consult [Attiya/Welch Alg. 3.1])

1: Each node $v$ does the following:
2: Initially all nodes are active. \{ all nodes may still become leaders \}
3: Whenever a node $v$ sees a message $w$ with $w > v$, then $v$ decides to not be a leader and becomes passive.
4: Active nodes search in an exponentially growing neighborhood (clockwise and counterclockwise) for nodes with higher identifiers, by sending out probe messages. A probe message includes the ID of the original sender, a bit whether the sender can still become a leader, and a time-to-live number (TTL). The first probe message sent by node $v$ includes a TTL of 1.
5: Nodes (active or passive) receiving a probe message decrement the TTL and forward the message to the next neighbor; if their ID is larger than the one in the message, they set the leader bit to zero, as the probing node does not have the maximum ID. If the TTL is zero, probe messages are returned to the sender using a reply message. The reply message contains the ID of the receiver (the original sender of the probe message) and the leader-bit. Reply messages are forwarded by all nodes until they reach the receiver.
6: Upon receiving the reply message: If there was no node with higher ID in the search area (indicated by the bit in the reply message), the TTL is doubled and two new probe messages are sent (again to the two neighbors). If there was a better candidate in the search area, then the node becomes passive.
7: If a node $v$ receives its own probe message (not a reply) $v$ decides to be the leader.

Remarks:

- This algorithm is asynchronous and uniform as well.
- The question may arise whether one can design an algorithm with an even lower message complexity. We answer this question in the next section.

2.4 Lower Bounds

Lower bounds in distributed computing are often easier than in the standard centralized (random access machine, RAM) model because one can argue about messages that need to be exchanged. In this section we present a first lower bound. We show that Algorithm 2 is asymptotically optimal.

Definition 2.14 (Execution). An execution of a distributed algorithm is a list of events, sorted by time. An event is a record (time, node, type, message), where type is "send" or "receive".

Remarks:

- We assume throughout this course that no two events happen at exactly the same time (or one can break ties arbitrarily).
- An execution of an asynchronous algorithm is generally not only determined by the algorithm but also by a “god-like” scheduler. If more than one message is in transit, the scheduler can choose which one arrives first.
If two messages are transmitted over the same directed edge, then it is sometimes required that the message first transmitted will also be received first (“FIFO”).

For our lower bound, we assume the following model:

- We are given an asynchronous ring, where nodes may wake up at arbitrary times (but at the latest when receiving the first message).
- We only accept uniform algorithms where the node with the maximum identifier can be the leader. Additionally, every node that is not the leader must know the identity of the leader. These two requirements can be dropped when using a more complicated proof; however, this is beyond the scope of this course.
- During the proof we will “play god” and specify which message in transmission arrives next in the execution. We respect the FIFO conditions for links.

**Definition 2.15 (Open Schedule).** A schedule is an execution chosen by the scheduler. A schedule for a ring is open if there is an open edge in the ring. An open (undirected) edge is an edge where no message traversing the edge has been received so far.

The proof of the lower bound is by induction. First we show the base case:

**Lemma 2.16.** Given a ring $R$ with two nodes, we can construct an open schedule in which at least one message is received. The nodes cannot distinguish this schedule from one on a larger ring with all other nodes being where the open edge is.

Proof: Let the two nodes be $u$ and $v$ with $u < v$. Node $u$ must learn the identity of node $v$, thus receive at least one message. We stop the execution of the algorithm as soon as the first message is received. (If the first message is received by $v$, bad luck for the algorithm!) Then the other edge in the ring (on which the received message was not transmitted) is open. Since the algorithm needs to be uniform, maybe the open edge is not really an edge at all, nobody can tell. We could use this to glue two rings together, by breaking up this imaginary open edge and connect two rings by two edges.

**Lemma 2.17.** By gluing together two rings of size $n/2$ for which we have open schedules, we can construct an open schedule on a ring of size $n$. If $M(n/2)$ denotes the number of messages already received in each of these schedules, at least $2M(n/2) + n/4$ messages have to be exchanged in order to solve leader election.

Proof by induction: We divide the ring into two sub-rings $R_1$ and $R_2$ of size $n/2$. These subrings cannot be distinguished from rings with $n/2$ nodes if no messages are received from “outsiders”. We can ensure this by not scheduling such messages until we want to. Note that executing both given open schedules on $R_1$ and $R_2$ “in parallel” is possible because we control not only the scheduling of the messages, but also when nodes wake up. By doing so, we make sure that $2M(n/2)$ messages are sent before the nodes in $R_1$ and $R_2$ learn anything of each other!

Without loss of generality, $R_1$ contains the maximum identifier. Hence, each node in $R_2$ must learn the identity of the maximum identifier, thus at least $n/2$ additional messages must be received. The only problem is that we cannot connect the two subrings with both edges since the new ring needs to remain open. Thus, only messages over one of the edges can be received. We “play god” and look into the future: we check what happens when we close only one of these connecting edges. With the argument that $n/2$ new messages must be received, we know that there is at least one
edge that will produce at least \( n/4 \) additional messages when being scheduled. (These messages may not be sent over the closed link, but they are \textit{caused} by a message over this link. They cannot involve any message along the other (open) edge at distance \( n/2 \).) We schedule this edge and the resulting \( n/4 \) messages, and leave the other open.

**Lemma 2.18.** Any uniform leader election algorithm for asynchronous rings has at least message complexity \( M(n) \geq \frac{n}{4}(\log n + 1) \).

Proof by induction: For simplicity we assume \( n \) being a power of 2. The base case \( n = 2 \) works because of Lemma 2.16 which implies that \( M(2) \geq 1 = \frac{2}{4}(\log 2 + 1) \).

For the induction step, using Lemma 2.17 and the induction hypothesis we have

\[
M(n) = 2 \cdot M\left(\frac{n}{2}\right) + \frac{n}{4} \\
\geq 2 \cdot \left(\frac{n}{8}\left(\log \frac{n}{2} + 1\right)\right) + \frac{n}{4} \\
= \frac{n}{4} \log n + \frac{n}{4} = \frac{n}{4} (\log n + 1) .
\]

\( \square \)

**Remarks:**

- To hide the ugly constants we use the “big Omega” notation, the lower bound equivalent of \( O() \). A function \( f \) is in \( \Omega(g) \) if there are constants \( x_0 \) and \( c > 0 \) such that \( |f(x)| \geq c|g(x)| \) for all \( x \geq x_0 \). Again we refer to standard text books for a formal definition. Rewriting Lemma 2.18 we get:

**Theorem 2.19 (Asynchronous Leader Election Lower Bound).** Any uniform leader election algorithm for asynchronous rings has \( \Omega(n \log n) \) message complexity.

## 2.5 Synchronous Ring

The lower bound relied on delaying messages for a very long time. Since this is impossible in the synchronous model, we might get a better message complexity in this case. The basic idea is very simple: In the synchronous model, not receiving a message is information as well! First we make some additional assumptions:

- We assume that the algorithm is non-uniform (i.e., the ring size \( n \) is known).
- We assume that every node starts at the same time.
- The node with the minimum identifier becomes the leader; identifiers are integers.

**Remarks:**

- Message complexity is indeed \( n \).
- But the time complexity is huge! If \( m \) is the minimum identifier it is \( m \cdot n \).
- The synchronous start and the non-uniformity assumptions can be dropped by using a wake-up technique (upon receiving a wake-up message, wake up your clockwise neighbors) and by letting messages travel slowly.
Algorithm 3 Synchronous Leader Election

1. Each node \( v \) concurrently executes the following code:
2. The algorithm operates in synchronous phases. Each phase consists of \( n \) time steps. Node \( v \) counts phases, starting with 0.
3. if phase = \( v \) and \( v \) did not yet receive a message then
4. \( v \) decides to be the leader
5. \( v \) sends the message "\( v \) is leader" around the ring
6. end if

- There are several lower bounds for the synchronous model: comparison-based algorithms or algorithms where the time complexity cannot be a function of the identifiers have message complexity \( \Omega(n \log n) \) as well.
- In general graphs efficient leader election may be tricky. While time-optimal leader election can be done by parallel flooding-echo (see next chapter), bounding the message complexity is generally more difficult.
Chapter 3

Tree Algorithms

In this chapter we learn a few basic algorithms on trees, and how to construct trees in
the first place so that we can run these (and other) algorithms. The good news is that
these algorithms have many applications, the bad news is that this chapter is a bit on
the simple side. But maybe that’s not really bad news?!

3.1 Broadcast

Definition 3.1 (Broadcast). A broadcast operation is initiated by a single processor,
the source. The source wants to send a message to all other nodes in the system.

Definition 3.2 (Distance, Radius, Diameter). The distance between two nodes $u$ and
$v$ in an undirected graph $G$ is the number of hops of a minimum path between $u$ and
$v$. The radius of a node $u$ is the maximum distance between $u$ and any other node in
the graph. The radius of a graph is the minimum radius of any node in the graph. The
diameter of a graph is the maximum distance between two arbitrary nodes.

Remarks:

- Clearly there is a close relation between the radius $R$ and the diameter $D$ of a
  graph, such as $R \leq D \leq 2R$.

- The world is often fascinated by graphs with a small radius. For example,
  movie fanatics study the who-acted-with-whom-in-the-same-movie graph. For
  this graph it has long been believed that the actor Kevin Bacon has a particularly
  small radius. The number of hops from Bacon even got a name, the Bacon Num-
  ber. In the meantime, however, it has been shown that there are “better” centers
  in the Hollywood universe, such as Sean Connery, Christopher Lee, Rod Steiger,
  Gene Hackman, or Michael Caine. The center of other social networks has also
  been explored, Paul Erdős for instance is well known in the math community.

Theorem 3.3 (Broadcast Lower Bound). The message complexity of broadcast is at
least $n - 1$. The source’s radius is a lower bound for the time complexity.

Proof: Every node must receive the message.
Remarks:

- You can use a pre-computed spanning tree to do broadcast with tight message complexity. If the spanning tree is a breadth-first search spanning tree (for a given source), then the time complexity is tight as well.

Definition 3.4 (Clean). A graph (network) is clean if the nodes do not know the topology of the graph.

Theorem 3.5 (Clean Broadcast Lower Bound). For a clean network, the number of edges is a lower bound for the broadcast message complexity.

Proof: If you do not try every edge, you might miss a whole part of the graph behind it.

Remarks:

- This lower bound proof directly brings us to the well known flooding algorithm.

Algorithm 4 Flooding

1: The source (root) sends the message to all neighbors.
2: Each other node $v$ upon receiving the message the first time forwards the message to all (other) neighbors.
3: Upon later receiving the message again (over other edges), a node can discard the message.

Remarks:

- If node $v$ receives the message first from node $u$, then node $v$ calls node $u$ parent. This parent relation defines a spanning tree $T$. If the flooding algorithm is executed in a synchronous system, then $T$ is a breadth-first search spanning tree (with respect to the root).
- More interestingly, also in asynchronous systems the flooding algorithm terminates after $R$ time units, $R$ being the radius of the source. However, the constructed spanning tree may not be a breadth-first search spanning tree.

3.2 Convergecast

Convergecast is the same as broadcast, just reversed: Instead of a root sending a message to all other nodes, all other nodes send information to a root. The simplest convergecast algorithm is the echo algorithm:

Algorithm 5 Echo

Require: This algorithm is initiated at the leaves.
1: A leave sends a message to its parent.
2: If an inner node has received a message from each child, it sends a message to the parent.
3.3. BFS TREE CONSTRUCTION

Remarks:

- Usually the echo algorithm is paired with the flooding algorithm, which is used to let the leaves know that they should start the echo process; this is known as flooding/echo.

- One can use convergecast for termination detection, for example. If a root wants to know whether all nodes in the system have finished some task, it initiates a flooding/echo; the message in the echo algorithm then means “This subtree has finished the task.”

- Message complexity of the echo algorithm is $n - 1$, but together with flooding it is $O(m)$, where $m = |E|$ is the number of edges in the graph.

- The time complexity of the echo algorithm is determined by the depth of the spanning tree (i.e., the radius of the root within the tree) generated by the flooding algorithm.

- The flooding/echo algorithm can do much more than collecting acknowledgements from subtrees. One can for instance use it to compute the number of nodes in the system, or the maximum ID (for leader election), or the sum of all values stored in the system, or a route-disjoint matching.

- Moreover, by combining results one can compute even fancier aggregations, e.g., with the number of nodes and the sum one can compute the average. With the average one can compute the standard deviation. And so on . . .

3.3 BFS Tree Construction

In synchronous systems the flooding algorithm is a simple yet efficient method to construct a breadth-first search (BFS) spanning tree. However, in asynchronous systems the spanning tree constructed by the flooding algorithm may be far from BFS. In this section, we implement two classic BFS constructions—Dijkstra and Bellman-Ford—as asynchronous algorithms.

We start with the Dijkstra algorithm. The basic idea is to always add the “closest” node to the existing part of the BFS tree. We need to parallelize this idea by developing the BFS tree layer by layer:

**Theorem 3.6 (Analysis of Algorithm 6).** The time complexity of Algorithm 6 is $O(D^2)$, the message complexity is $O(m + nD)$, where $D$ is the diameter of the graph, $n$ the number of nodes, and $m$ the number of edges.

Proof: A broadcast/echo algorithm in $T_p$ needs at most time $2D$. Finding new neighbors at the leaves costs 2 time units. Since the BFS tree height is bounded by the diameter, we have $D$ phases, giving a total time complexity of $O(D^2)$. Each node participating in broadcast/echo only receives (broadcasts) at most 1 message and sends (echoes) at most once. Since there are $D$ phases, the cost is bounded by $O(nD)$. On each edge there are at most 2 “join” messages. Replies to a “join” request are answered by 1 “ACK” or “NACK”, which means that we have at most 4 additional messages per edge. Therefore the message complexity is $O(m + nD)$. 


CHAPTER 3. TREE ALGORITHMS

Algorithm 6 Dijkstra BFS
1: The algorithm proceeds in phases. In phase $p$ the nodes with distance $p$ to the root are detected. Let $T_p$ be the tree in phase $p$. We start with $T_1$ which is the root plus all direct neighbors of the root. We start with phase $p = 1$:
2: repeat
3: The root starts phase $p$ by broadcasting “start $p$” within $T_p$.
4: When receiving “start $p$” a leaf node $u$ of $T_p$ (that is, a node that was newly discovered in the last phase) sends a “join $p + 1$” message to all quiet neighbors. (A neighbor $v$ is quiet if $u$ has not yet “talked” to $v$.)
5: A node $v$ receiving the first “join $p + 1$” message replies with “ACK” and becomes a leaf of the tree $T_{p+1}$.
6: A node $v$ receiving any further “join” message replies with “NACK”.
7: The leaves of $T_p$ collect all the answers of their neighbors; then the leaves start an echo algorithm back to the root.
8: When the echo process terminates at the root, the root increments the phase
9: until there was no new node detected

Remarks:

- The time complexity is not very exciting, so let’s try Bellman-Ford!

The basic idea of Bellman-Ford is even simpler, and heavily used in the Internet, as it is a basic version of the omnipresent border gateway protocol (BGP). The idea is to simply keep the distance to the root accurate. If a neighbor has found a better route to the root, a node might also need to update its distance.

Algorithm 7 Bellman-Ford BFS
1: Each node $u$ stores an integer $d_u$ which corresponds to the distance from $u$ to the root. Initially $d_{root} = 0$, and $d_u = \infty$ for every other node $u$.
2: The root starts the algorithm by sending “1” to all neighbors.
3: if a node $u$ receives a message “$y$” with $y < d_u$ from a neighbor $v$ then
4: node $u$ sets $d_u := y$
5: node $u$ sends “$y + 1$” to all neighbors (except $v$)
6: end if

Theorem 3.7 (Analysis of Algorithm 7). The time complexity of Algorithm 7 is $O(D)$, the message complexity is $O(nm)$, where $D, n, m$ are defined as in Theorem 3.6.

Proof: We can prove the time complexity by induction. We claim that a node at distance $d$ from the root has received a message “$d$” by time $d$. The root knows by time 0 that it is the root. A node $v$ at distance $d$ has a neighbor $u$ at distance $d - 1$. Node $u$ by induction sends a message “$d$” to $v$ at time $d - 1$ or before, which is then received by $v$ at time $d$ or before. Message complexity is easier: A node can reduce its distance at most $n - 1$ times; each of these times it sends a message to all its neighbors. If all nodes do this we have $O(nm)$ messages.
3.4 MST Construction

There are several types of spanning trees, each serving a different purpose. A particularly interesting spanning tree is the minimum spanning tree (MST). The MST only makes sense on weighted graphs, hence in this section we assume that each edge $e$ is assigned a weight $\omega_e$.

**Definition 3.8 (MST).** Given a weighted graph $G = (V, E, \omega)$, the MST of $G$ is a spanning tree $T$ minimizing $\omega(T)$, where $\omega(G') = \sum_{e \in G'} \omega_e$ for any subgraph $G' \subseteq G$.

Remarks:
- In the following we assume that no two edges of the graph have the same weight. This simplifies the problem as it makes the MST unique; however, this simplification is not essential as one can always break ties by adding the IDs of adjacent vertices to the weight.
- Obviously we are interested in computing the MST in a distributed way. For this we use a well-known lemma:

**Definition 3.9 (Blue Edges).** Let $T$ be a spanning tree of the weighted graph $G$ and $T' \subseteq T$ a subgraph of $T$ (also called a fragment). Edge $e = (u, v)$ is an outgoing edge of $T'$ if $u \in T'$ and $v \notin T'$ (or vice versa). The minimum weight outgoing edge $b(T')$ is the so-called blue edge of $T'$.

**Lemma 3.10.** For a given weighted graph $G$ (such that no two weights are the same), let $T$ denote the MST, and $T'$ be a fragment of $T$. Then the blue edge of $T'$ is also part of $T$, i.e., $T' \cup b(T') \subseteq T$.

Proof: For the sake of contradiction, suppose that in the MST $T$ there is edge $e \neq b(T')$ connecting $T'$ with the remainder of $T$. Adding the blue edge $b(T')$ to the MST $T$ we get a cycle including both $e$ and $b(T')$. If we remove $e$ from this cycle we still have a spanning tree, and since by the definition of the blue edge $\omega_e > \omega_{b(T')}$, the weight of that new spanning tree is less than than the weight of $T$. We have a contradiction.

Remarks:
- In other words, the blue edges seem to be the key to a distributed algorithm for the MST problem. Since every node itself is a fragment of the MST, every node directly has a blue edge! All we need to do is to grow these fragments! Essentially this is a distributed version of Kruskal’s sequential algorithm.
- At any given time the nodes of the graph are partitioned into fragments (rooted subtrees of the MST). Each fragment has a root, the ID of the fragment is the ID of its root. Each node knows its parent and its children in the fragment. The algorithm operates in phases. At the beginning of a phase, nodes know the IDs of the fragments of their neighbor nodes.
Algorithm 8 GHS (Gallager–Humblet–Spira)

1: Initially each node is the root of its own fragment. We proceed in phases:
2: repeat
3: All nodes learn the fragment IDs of their neighbors.
4: The root of each fragment uses flooding/echo in its fragment to determine the blue edge $b = (u, v)$ of the fragment.
5: The root sends a message to node $u$; while forwarding the message on the path from the root to node $u$ all parent-child relations are inverted [such that $u$ is the new temporary root of the fragment]
6: node $u$ sends a merge request over the blue edge $b = (u, v)$.
7: if node $v$ also sent a merge request over the same blue edge $b = (v, u)$ then
8: either $u$ or $v$ (whichever has the smaller ID) is the new fragment root
9: the blue edge $b$ is directed accordingly
10: else
11: node $v$ is the new parent of node $u$
12: end if
13: the newly elected root node informs all nodes in its fragment (again using flooding/echo) about its identity
14: until all nodes are in the same fragment (i.e., there is no outgoing edge)

Remarks:

- Algorithm 8 was stated in pseudo-code, with a few details not really explained. For instance, it may be that some fragments are much larger than others, and because of that some nodes may need to wait for others, e.g., if node $u$ needs to find out whether neighbor $v$ also wants to merge over the blue edge $b = (u, v)$. The good news is that all these details can be solved. We can for instance bound the asynchronicity by guaranteeing that nodes only start the new phase after the last phase is done, similarly to the phase-technique of Algorithm 6.

Theorem 3.11 (Analysis of Algorithm 8). The time complexity of Algorithm 8 is $O(n \log n)$, the message complexity is $O(m \log n)$.

Proof: Each phase mainly consists of two flooding/echo processes. In general, the cost of flooding/echo on a tree is $O(D)$ time and $O(n)$ messages. However, the diameter $D$ of the fragments may turn out to be not related to the diameter of the graph because the MST may meander, hence it really is $O(n)$ time. In addition, in the first step of each phase, nodes need to learn the fragment ID of their neighbors; this can be done in 2 steps but costs $O(m)$ messages. There are a few more steps, but they are cheap. Altogether a phase costs $O(n)$ time and $O(m)$ messages. So we only have to figure out the number of phases: Initially all fragments are single nodes and hence have size 1. In a later phase, each fragment merges with at least one other fragment, that is, the size of the smallest fragment at least doubles. In other words, we have at most $\log n$ phases. The theorem follows directly.

Remarks:

- Algorithm 8 is called “GHS” after Gallager, Humblet, and Spira, three pioneers in distributed computing. Despite being quite simple the algorithm won the prestigious Edsger W. Dijkstra Prize in Distributed Computing in 2004, among other reasons because it was one of the first (1983) non-trivial asynchronous distributed algorithms. As such it can be seen as one of the seeds of this research area.
3.4. MST CONSTRUCTION

- We presented a simplified version of GHS. The original paper by Gallager et al. featured an improved message complexity of $O(m + n \log n)$.

- In 1987, Awerbuch managed to further improve the GHS algorithm to get $O(n)$ time and $O(m + n \log n)$ message complexity, both asymptotically optimal.

- The GHS algorithm can be applied in different ways. GHS for instance directly solves leader election in general graphs: The leader is simply the last surviving root!
Chapter 4

Vertex Coloring

4.1 Introduction

Vertex coloring is an infamous graph theory problem. Vertex coloring does have quite a few practical applications, for example in the area of wireless networks where coloring is the foundation of so-called TDMA MAC protocols. Generally speaking, vertex coloring is used as a means to break symmetries, one of the main themes in distributed computing. In this chapter we will not really talk about vertex coloring applications but treat the problem abstractly. At the end of the class you probably learned the fastest (but not constant!) algorithm ever! Let us start with some simple definitions and observations.

Problem 4.1 (Vertex Coloring). Given an undirected graph $G = (V, E)$, assign a color $c_u$ to each vertex $u \in V$ such that the following holds: $e = (v, w) \in E \Rightarrow c_v \neq c_w$.

Remarks:

- Throughout this course, we use the terms vertex and node interchangeably.

- The application often asks us to use few colors! In a TDMA MAC protocol, for example, less colors immediately imply higher throughput. However, in distributed computing we are often happy with a solution which is suboptimal. There is a tradeoff between the optimality of a solution (efficacy), and the work/time needed to compute the solution (efficiency).

Figure 4.1: 3-colorable graph with a valid coloring.

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**Assumption 4.2** (Node Identifiers). *Each node has a unique identifier, e.g., its IP address. We usually assume that each identifier consists of only \( \log n \) bits if the system has \( n \) nodes.*

**Remarks:**

- Sometimes we might even assume that the nodes exactly have identifiers \( 1, \ldots, n \).
- It is easy to see that node identifiers (as defined in Assumption 4.2) solve the coloring problem 4.1, but not very well (essentially requiring \( n \) colors). How many colors are needed at least is a well-studied problem.

**Definition 4.3** (Chromatic Number). *Given an undirected Graph \( G = (V, E) \), the chromatic number \( \chi(G) \) is the minimum number of colors to solve Problem 4.1.*

To get a better understanding of the vertex coloring problem, let us first look at a simple non-distributed (“centralized”) vertex coloring algorithm:

**Algorithm 9** Greedy Sequential

1: while \( \exists \) uncolored vertex \( v \) do
2: color \( v \) with the minimal color (number) that does not conflict with the already colored neighbors
3: end while

**Definition 4.4** (Degree). *The number of neighbors of a vertex \( v \), denoted by \( \delta(v) \), is called the degree of \( v \). The maximum degree vertex in a graph \( G \) defines the graph degree \( \Delta(G) = \Delta \).*

**Theorem 4.5** (Analysis of Algorithm 9). *The algorithm is correct and terminates in \( n \) “steps”. The algorithm uses \( \Delta + 1 \) colors.*

Proof: Correctness and termination are straightforward. Since each node has at most \( \Delta \) neighbors, there is always at least one color free in the range \( \{1, \ldots, \Delta + 1\} \).

**Remarks:**

- For many graphs coloring can be done with much less than \( \Delta + 1 \) colors.
- This algorithm is not distributed at all; only one processor is active at a time. Still, maybe we can use the simple idea of Algorithm 9 to define a distributed coloring subroutine that may come in handy later.

Now we are ready to study distributed algorithms for this problem. The following procedure can be executed by every vertex \( v \) in a distributed coloring algorithm. The goal of this subroutine is to improve a given initial coloring.

**Procedure 10** First Free

**Require:** Node Coloring {e.g., node IDs as defined in Assumption 4.2}

Give \( v \) the smallest admissible color {i.e., the smallest node color not used by any neighbor}
4.2. COLORING TREES

Remarks:

- With this subroutine we have to make sure that two adjacent vertices are not colored at the same time. Otherwise, the neighbors may at the same time conclude that some small color \( c \) is still available in their neighborhood, and then at the same time decide to choose this color \( c \).

Algorithm 11 Reduce

1: Assume that initially all nodes have ID’s (Assumption 4.2)
2: Each node \( v \) executes the following code
3: node \( v \) sends its ID to all neighbors
4: node \( v \) receives IDs of neighbors
5: while node \( v \) has an uncolored neighbor with higher ID do
6: node \( v \) sends “undecided” to all neighbors
7: node \( v \) receives new decisions from neighbors
8: end while
9: node \( v \) chooses a free color using subroutine First Free (Procedure 10)
10: node \( v \) informs all its neighbors about its choice

Figure 4.2: Vertex 100 receives the lowest possible color.

Theorem 4.6 (Analysis of Algorithm 11). Algorithm 11 is correct and has time complexity \( n \). The algorithm uses \( \Delta + 1 \) colors.

Remarks:

- Quite trivial, but also quite slow.
- However, it seems difficult to come up with a fast algorithm.
- Maybe it’s better to first study a simple special case, a tree, and then go from there.

4.2 Coloring Trees

Lemma 4.7. \( \chi(Tree) \leq 2 \)

Constructive Proof: If the distance of a node to the root is odd (even), color it 1 (0). An odd node has only even neighbors and vice versa. If we assume that each node knows its parent (root has no parent) and children in a tree, this constructive proof gives a very simple algorithm:
Algorithm 12 Slow Tree Coloring

1: Color the root 0, root sends 0 to its children
2: Each node \( v \) concurrently executes the following code:
3: if node \( v \) receives a message \( x \) (from parent) then
4: node \( v \) chooses color \( c_v = 1 - x \)
5: node \( v \) sends \( c_v \) to its children (all neighbors except parent)
6: end if

Remarks:

- With the proof of Lemma 4.7, Algorithm 12 is correct.
- How can we determine a root in a tree if it is not already given? We will figure that out later.
- The time complexity of the algorithm is the height of the tree.
- If the root was chosen unfortunately, and the tree has a degenerated topology, the time complexity may be up to \( n \), the number of nodes.
- Also, this algorithm does not need to be synchronous . . . !

Theorem 4.8 (Analysis of Algorithm 12). Algorithm 12 is correct. If each node knows its parent and its children, the (asynchronous) time complexity is the tree height which is bounded by the diameter of the tree; the message complexity is \( n - 1 \) in a tree with \( n \) nodes.

Remarks:

- In this case the asynchronous time complexity is the same as the synchronous time complexity.
- Nice trees, e.g. balanced binary trees, have logarithmic height, that is we have a logarithmic time complexity.
- This algorithm is not very exciting. Can we do better than logarithmic?!?

The following algorithm terminates in \( \log^* n \) time. Log-Star?! That’s the number of logarithms (to the base 2) you need to take to get down to at least 2, starting with \( n \):

Definition 4.9 (Log-Star).
\[
\forall x \leq 2 : \log^* x := 1 \quad \forall x > 2 : \log^* x := 1 + \log^*(\log x)
\]

Remarks:

- Log-star is an amazingly slowly growing function. Log-star of all the atoms in the observable universe (estimated to be \( 10^{80} \)) is 5! There are functions which grow even more slowly, such as the inverse Ackermann function, however, the inverse Ackermann function of all the atoms is 4. So log-star increases indeed very slowly!
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Algorithm 13 “6-Color”

1: Assume that initially the vertices are legally colored. Using Assumption 4.2 each label only has \( \log n \) bits.
2: The root assigns itself the label 0.
3: Each other node \( v \) executes the following code (synchronously in parallel):
4: send \( c_v \) to all children
5: repeat
6: receive \( c_p \) from parent
7: interpret \( c_v \) and \( c_p \) as little-endian bit-strings: \( c(1), \ldots, c(1), c(0) \)
8: let \( i \) be the smallest index where \( c_v \) and \( c_p \) differ
9: the new label is \( i \) (as bitstring) followed by the bit \( c_v(i) \) itself
10: send \( c_v \) to all children
11: until \( c_w \in \{0, \ldots, 5\} \) for all nodes \( w \)

Here is the idea of the algorithm: We start with color labels that have \( \log n \) bits. In each synchronous round we compute a new label with exponentially smaller size than the previous label, still guaranteeing to have a valid vertex coloring! But how are we going to do that?

Example:

Algorithm 13 executed on the following part of a tree:

| Grand-parent | 0010110000 \( \rightarrow \) 10010 \( \rightarrow \) \ldots |
| Parent       | 1010010000 \( \rightarrow \) 01010 \( \rightarrow \) 111   |
| Child        | 0110010000 \( \rightarrow \) 10001 \( \rightarrow \) 001   |

Theorem 4.10 (Analysis of Algorithm 13). Algorithm 13 terminates in \( \log^* n \) time.

Proof: A detailed proof is, e.g., in [Peleg 7.3]. In class we do a sketch of the proof.

Remarks:

- Colors 11* (in binary notation, i.e., 6 or 7 in decimal notation) will not be chosen, because the node will then do another round. This gives a total of 6 colors (i.e., colors 0, \ldots, 5).

- Can one reduce the number of colors in only constant steps? Note that algorithm 11 does not work (since the degree of a node can be much higher than 6)! For fewer colors we need to have siblings monochromatic!

- Before we explore this problem we should probably have a second look at the end game of the algorithm, the UNTIL statement. Is this algorithm truly local?!
Let’s discuss!

Algorithm 14 Shift Down

1: Root chooses a new (different) color from \( \{0, 1, 2\} \)
2: Each other node \( v \) concurrently executes the following code:
3: Recolor \( v \) with the color of parent
Lemma 4.11 (Analysis of Algorithm 14). Algorithm 14 preserves coloring legality; also siblings are monochromatic.

Now Algorithm 11 (Reduce) can be used to reduce the number of used colors from six to three.

Algorithm 15 Six-2-Three
1: Each node \( v \) concurrently executes the following code:
2: Run Algorithm 13 for \( \log^* n \) rounds.
3: for \( x = 5, 4, 3 \) do
4: Perform subroutine Shift down (Algorithm 14)
5: if \( c_v = x \) then
6: choose new color \( c_v \in \{0, 1, 2\} \) using subroutine First Free (Algorithm 10)
7: end if
8: end for

Theorem 4.12 (Analysis of Algorithm 15). Algorithm 15 colors a tree with three colors in time \( O(\log^* n) \).

Remarks:

- The term \( O() \) used in Theorem 4.10 is called “big O” and is often used in distributed computing. Roughly speaking, \( O(f) \) means “in the order of \( f \), ignoring constant factors and smaller additive terms.” More formally, for two functions \( f \) and \( g \), it holds that \( f \in O(g) \) if there are constants \( x_0 \) and \( c \) so that \( |f(x)| \leq c|g(x)| \) for all \( x \geq x_0 \). For an elaborate discussion on the big O notation we refer to other introductory math or computer science classes.

- As one can easily prove, a fast tree-coloring with only 2 colors is more than exponentially more expensive than coloring with 3 colors. In a tree degenerated to a list, nodes far away need to figure out whether they are an even or odd number of hops away from each other in order to get a 2-coloring. To do that one has to send a message to these nodes. This costs time linear in the number of nodes.

- Also other lower bounds have been proved, e.g., any algorithm for 2-coloring the \( d \)-regular tree of radius \( r \) which runs in time at most \( 2r/3 \) requires at least \( \Omega(\sqrt{d}) \) colors.

- The idea of this algorithm can be generalized, e.g., to a ring topology. Also a general graph with constant degree \( \Delta \) can be colored with \( \Delta + 1 \) colors in \( O(\log^* n) \) time. The idea is as follows: In each step, a node compares its label to each of its neighbors, constructing a logarithmic difference-tag as in 6-color (Algorithm 13). Then the new label is the concatenation of all the difference-tags. For constant degree \( \Delta \), this gives a 3\( \Delta \)-label in \( O(\log^* n) \) steps. Algorithm 11 then reduces the number of colors to \( \Delta + 1 \) in \( 2^{3\Delta} \) (this is still a constant for constant \( \Delta \)!) steps.

- Recently, researchers have proposed other methods to break down long ID’s for log-star algorithms. With these new techniques, one is able to solve other problems, e.g., a maximal independent set in bounded growth graphs in \( O(\log^* n) \) time. These techniques go beyond the scope of this course.
4.2. COLORING TREES

Figure 4.3: Possible execution of Algorithm 15.

- Unfortunately, coloring a general graph is not yet possible with this technique. We will see another technique for that in Chapter 5. With this technique it is possible to color a general graph with \( \Delta + 1 \) colors in \( O(\log n) \) time.

- A lower bound by Linial shows that many of these log-star algorithms are asymptotically (up to constant factors) optimal. This lower bound uses an interesting technique. However, because of the one-topic-per-class policy we cannot look at it today.
Chapter 5

Maximal Independent Set

In this chapter we present a highlight of this course, a fast maximal independent set (MIS) algorithm. The algorithm is the first randomized algorithm that we study in this class. In distributed computing, randomization is a powerful and therefore omnipresent concept, as it allows for relatively simple yet efficient algorithms. As such the studied algorithm is archetypal.

A MIS is a basic building block in distributed computing, some other problems pretty much follow directly from the MIS problem. At the end of this chapter, we will give two examples: matching and vertex coloring (see Chapter 4).

5.1 MIS

Definition 5.1 (Independent Set). Given an undirected Graph $G = (V, E)$ an independent set is a subset of nodes $U \subseteq V$, such that no two nodes in $U$ are adjacent. An independent set is maximal if no node can be added without violating independence. An independent set of maximum cardinality is called maximum.

Figure 5.1: Example graph with 1) a maximal independent set (MIS) and 2) a maximum independent set (MaxIS).
CHAPTER 5. MAXIMAL INDEPENDENT SET

Remarks:

- Computing a maximum independent set (MaxIS) is a notoriously difficult problem. It is equivalent to maximum clique on the complementary graph. Both problems are NP-hard, in fact not approximable within \( n^{\frac{1}{2}-\epsilon} \).

- In this course we concentrate on the maximal independent set (MIS) problem. Please note that MIS and MaxIS can be quite different, indeed e.g. on a star graph the MIS is \( \Theta(n) \) smaller than the MaxIS (cf. Figure 5.1).

- Computing a MIS sequentially is trivial: Scan the nodes in arbitrary order. If a node \( u \) does not violate independence, add \( u \) to the MIS. If \( u \) violates independence, discard \( u \). So the only question is how to compute a MIS in a distributed way.

```plaintext
Algorithm 16 Slow MIS

Require: Node IDs

Every node \( v \) executes the following code:
1: if all neighbors of \( v \) with larger identifiers have decided not to join the MIS then
2: \( v \) decides to join the MIS
3: end if
```

Remarks:

- Not surprisingly the slow algorithm is not better than the sequential algorithm in the worst case, because there might be one single point of activity at any time. Formally:

**Theorem 5.2** (Analysis of Algorithm 16). Algorithm 16 features a time complexity of \( O(n) \) and a message complexity of \( O(m) \).

Remarks:

- This is not very exciting.

- There is a relation between independent sets and node coloring (Chapter 4), since each color class is an independent set, however, not necessarily a MIS. Still, starting with a coloring, one can easily derive a MIS algorithm: We first choose all nodes of the first color. Then, for each additional color we add “in parallel” (without conflict) as many nodes as possible. Thus the following corollary holds:

**Corollary 5.3.** Given a coloring algorithm that needs \( C \) colors and runs in time \( T \), we can construct a MIS in time \( C + T \).

Remarks:

- Using Theorem 4.12 and Corollary 5.3 we get a distributed deterministic MIS algorithm for trees (and for bounded degree graphs) with time complexity \( O(\log^* n) \).

- With a lower bound argument one can show that this deterministic MIS algorithm for rings is asymptotically optimal.
5.2. FAST MIS FROM 1986

There have been attempts to extend Algorithm 13 to more general graphs, however, so far without much success. Below we present a radically different approach that uses randomization. Please note that the algorithm and the analysis below is not identical with the algorithm in Peleg’s book.

5.2 Fast MIS from 1986

Algorithm 17 Fast MIS
The algorithm operates in synchronous rounds, grouped into phases.
A single phase is as follows:
1) Each node $v$ marks itself with probability $\frac{1}{2d(v)}$, where $d(v)$ is the current degree of $v$.
2) If no higher degree neighbor of $v$ is also marked, node $v$ joins the MIS. If a higher degree neighbor of $v$ is marked, node $v$ unmarks itself again. (If the neighbors have the same degree, ties are broken arbitrarily, e.g., by identifier).
3) Delete all nodes that joined the MIS and their neighbors, as they cannot join the MIS anymore.

Remarks:
• Correctness in the sense that the algorithm produces an independent set is relatively simple: Steps 1 and 2 make sure that if a node $v$ joins the MIS, then $v$‘s neighbors do not join the MIS at the same time. Step 3 makes sure that $v$‘s neighbors will never join the MIS.
• Likewise the algorithm eventually produces a MIS, because the node with the highest degree will mark itself at some point in Step 1.
• So the only remaining question is how fast the algorithm terminates. To understand this, we need to dig a bit deeper.

Lemma 5.4 (Joining MIS). A node $v$ joins the MIS in step 2 with probability $p \geq \frac{1}{4d(v)}$.

Proof: Let $M$ be the set of marked nodes in step 1. Let $H(v)$ be the set of neighbors of $v$ with higher degree, or same degree and higher identifier. Using independence of the random choices of $v$ and nodes in $H(v)$ in Step 1 we get

$$
P[v \notin MIS|v \in M] = P[\exists w \in H(v), w \in M|v \in M]
= P[\exists w \in H(v), w \in M]
\leq \sum_{w \in H(v)} P[w \in M] = \sum_{w \in H(v)} \frac{1}{2d(w)}
\leq \sum_{w \in H(v)} \frac{1}{2d(v)} \leq \frac{d(v)}{2d(v)} = \frac{1}{2}.
$$

Then

$$
P[v \in MIS] = P[v \in MIS|v \in M] \cdot P[v \in M] \geq \frac{1}{2} \cdot \frac{1}{2d(v)}.
$$

□
CHAPTER 5. MAXIMAL INDEPENDENT SET

Lemma 5.5 (Good Nodes). A node $v$ is called good if

$$\sum_{w \in N(v)} \frac{1}{2d(w)} \geq \frac{1}{6}.$$ 

Otherwise we call $v$ a bad node. A good node will be removed in Step 3 with probability $p \geq \frac{1}{36}$.

Proof: Let node $v$ be good. Intuitively, good nodes have lots of low-degree neighbors, thus chances are high that one of them goes into the independent set, in which case $v$ will be removed in step 3 of the algorithm.

If there is a neighbor $w \in N(v)$ with degree at most 2 we are done: With Lemma 5.4 the probability that node $w$ joins the MIS is at least $\frac{1}{8}$, and our good node will be removed in Step 3.

So all we need to worry about is that all neighbors have at least degree 3: For any neighbor $w$ of $v$ we have $\frac{1}{2d(w)} \leq \frac{1}{3}$. Since $\sum_{w \in N(v)} \frac{1}{2d(w)} \geq \frac{1}{6}$ there is a subset of neighbors $S \subseteq N(v)$ such that $\frac{1}{6} \leq \sum_{w \in S} \frac{1}{2d(w)} \leq \frac{1}{3}$.

We can now bound the probability that node $v$ will be removed. Let therefore $R$ be the event of $v$ being removed. Again, if a neighbor of $v$ joins the MIS in Step 2, node $v$ will be removed in Step 3. We have

$$P[R] \geq P[\exists u \in S, u \in \text{MIS}] - \sum_{u, w \in S; u \neq w} P[u \in \text{MIS} \text{ and } w \in \text{MIS}].$$

For the last inequality we used the inclusion-exclusion principle truncated after the second order terms. Let $M$ again be the set of marked nodes after Step 1. Using $P[u \in M] \geq P[u \in \text{MIS}]$ we get

$$P[R] \geq \sum_{u \in S} P[u \in \text{MIS}] - \sum_{u, w \in S; u \neq w} P[u \in M \text{ and } w \in M]$$

$$\geq \sum_{u \in S} P[u \in \text{MIS}] - \sum_{u \in S} \sum_{w \in S} P[u \in M] \cdot P[w \in M]$$

$$\geq \sum_{u \in S} \frac{1}{4d(u)} - \sum_{w \in S} \sum_{u \in S} \frac{1}{2d(u)} \frac{1}{2d(w)}$$

$$\geq \sum_{u \in S} \frac{1}{2d(u)} \left( \frac{1}{2} - \sum_{w \in S} \frac{1}{2d(w)} \right) \geq \frac{1}{6} \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{1}{36}.$$

Remarks:

- We would be almost finished if we could prove that many nodes are good in each phase. Unfortunately this is not the case: In a star-graph, for instance, only a single node is good! We need to find a work-around.
Lemma 5.6 (Good Edges). An edge $e = (u, v)$ is called bad if both $u$ and $v$ are bad; else the edge is called good. The following holds: At any time at least half of the edges are good.

Proof: For the proof we construct a directed auxiliary graph: Direct each edge towards the higher degree node (if both nodes have the same degree direct it towards the higher identifier). Now we need a little helper lemma before we can continue with the proof.

Lemma 5.7. A bad node has outdegree at least twice its indegree.

Proof: For the sake of contradiction, assume that a bad node $v$ does not have outdegree at least twice its indegree. In other words, at least one third of the neighbor nodes (let’s call them $S$) have degree at most $d(v)$. But then

$$\sum_{w \in N(v)} \frac{1}{2d(w)} \geq \sum_{w \in S} \frac{1}{2d(v)} \geq \sum_{w \in S} \frac{1}{3d(v)} = \frac{1}{6}$$

which means $v$ is good, a contradiction. □

Continuing the proof of Lemma 5.6: According to Lemma 5.7 the number of edges directed into bad nodes is at most half the number of edges directed out of bad nodes. Thus, the number of edges directed into bad nodes is at most half the number of edges. Thus, at least half of the edges are directed into good nodes. Since these edges are not bad, they must be good.

Theorem 5.8 (Analysis of Algorithm 17). Algorithm 17 terminates in expected time $O(\log n)$.

Proof: With Lemma 5.5 a good node (and therefore a good edge!) will be deleted with constant probability. Since at least half of the edges are good (Lemma 5.6) a constant fraction of edges will be deleted in each phase.

More formally: With Lemmas 5.5 and 5.6 we know that at least half of the edges will be removed with probability at least $1/36$. Let $R$ be the number of edges to be removed. Using linearity of expectation we know that $E[R] \geq m/72$, $m$ being the total number of edges at the start of a phase. Now let $p := P[R \leq E[R]/2]$. Bounding the expectation yields

$$E[R] = \sum_{r} P[R = r] \cdot r \leq p \cdot E[R]/2 + (1 - p) \cdot m.$$ 

Solving for $p$ we get

$$p \leq \frac{m - E[R]}{m - E[R]/2} \leq \frac{m - E[R]/2}{m} \leq 1 - 1/144.$$

In other words, with probability at least $1/144$ at least $m/144$ edges are removed in a phase. After expected $O(\log m)$ phases all edges are deleted. Since $m \leq n^2$ and thus $O(\log m) = O(\log n)$ the Theorem follows. □
Remarks:

- With a bit of more math one can even show that Algorithm 17 terminates in time $O(\log n)$ "with high probability".

- The presented algorithm is a simplified version of an algorithm by Michael Luby, published 1986 in the SIAM Journal of Computing. Around the same time there have been a number of other papers dealing with the same or related problems, for instance by Alon, Babai, and Itai, or by Israeli and Itai. The analysis presented here takes elements of all these papers, and from other papers on distributed weighted matching. The analysis in the book by David Peleg is different, and only achieves $O(\log^2 n)$ time.

- Though not as incredibly fast as the $\log^*\cdot$-coloring algorithm for trees, this algorithm is very general. It works on any graph, needs no identifiers, and can easily be made asynchronous.

- Surprisingly, much later, there have been half a dozen more papers published, with much worse results!! In 2002, for instance, there was a paper with linear running time, improving on a 1994 paper with cubic running time, restricted to trees!

- In 2009, Métivier, Robson, Saheb-Djahromi and Zemmari found a slightly different (and simpler) way to compute a MIS in the same logarithmic time:

### 5.3 Fast MIS from 2009

**Algorithm 18 Fast MIS 2**

The algorithm operates in synchronous rounds, grouped into phases.

A single phase is as follows:

1) Each node $v$ chooses a random value $r(v) \in [0, 1]$ and sends it to its neighbors.
2) If $r(v) < r(w)$ for all neighbors $w \in N(v)$, node $v$ enters the MIS and informs its neighbors.
3) If $v$ or a neighbor of $v$ entered the MIS, $v$ terminates ($v$ and all edges adjacent to $v$ are removed from the graph), otherwise $v$ enters the next phase.

Remarks:

- Correctness in the sense that the algorithm produces an independent set is simple: Steps 1 and 2 make sure that if a node $v$ joins the MIS, then $v$’s neighbors do not join the MIS at the same time. Step 3 makes sure that $v$’s neighbors will never join the MIS.

- Likewise the algorithm eventually produces a MIS, because the node with the globally smallest value will always join the MIS, hence there is progress.

- So the only remaining question is how fast the algorithm terminates. To understand this, we need to dig a bit deeper.

- Our proof will rest on a simple, yet powerful observation about expected values of random variables that may not be independent:
5.3. FAST MIS FROM 2009

**Theorem 5.9** (Linearity of Expectation). Let $X_i, i = 1, \ldots, k$ denote random variables, then

$$E \left[ \sum_i X_i \right] = \sum_i E[X_i].$$

**Proof.** It is sufficient to prove $E[X + Y] = E[X] + E[Y]$ for two random variables $X$ and $Y$, because then the statement follows by induction. Since

$$P[(X, Y) = (x, y)] = P[X = x] \cdot P[Y = y|X = x]$$

we get that

$$E[X + Y] = \sum_{(X,Y)=(x,y)} P[(X, Y) = (x, y)] \cdot (x + y)$$

$$= \sum_{X=x} \sum_{Y=y} P[X = x] \cdot P[Y = y|X = x] \cdot x$$

$$+ \sum_{Y=y} \sum_{X=x} P[Y = y] \cdot P[X = x|Y = y] \cdot y$$

$$= \sum_{X=x} P[X = x] \cdot x + \sum_{Y=y} P[Y = y] \cdot y$$

$$= E[X] + E[Y].$$

**Remarks:**

- How can we prove that the algorithm only needs $O(\log n)$ phases in expectation? It would be great if this algorithm managed to remove a constant fraction of nodes in each phase. Unfortunately, it does not.

- Instead we will prove that the number of edges decreases quickly. Again, it would be great if any single edge was removed with constant probability in Step 3. But again, unfortunately, this is not the case.

- Maybe we can argue about the expected number of edges to be removed in one single phase? Let’s see: A node $v$ enters the MIS with probability $1/(d(v) + 1)$, where $d(v)$ is the degree of node $v$. By doing so, not only are $v$’s edges removed, but indeed all the edges of $v$’s neighbors as well – generally these are much more than $d(v)$ edges. So there is hope, but we need to be careful: If we do this the most naive way, we will count the same edge many times.

- How can we fix this? The nice observation is that it is enough to count just some of the removed edges. Given a new MIS node $v$ and a neighbor $w \in N(v)$, we count the edges only if $r(v) < r(x)$ for all $x \in N(w)$. This looks promising. In a star graph, for instance, only the smallest random value can be accounted for removing all the edges of the star.

**Lemma 5.10** (Edge Removal). In a single phase, we remove at least half of the edges in expectation.
This enables us to follow a bound on the expected running time of Algorithm 18. Algorithm 18 terminates of these edges. To get rid of all but two edges we need at least \(1\) phases in expectation. As \(v\) joins the MIS, all edges \((w, x)\) will be removed; there are \(d(w)\) of these edges.

In order to count the removed edges, we need to weigh events properly.

Whether we remove the edges adjacent to \(w\) because of event \((v \to w)\) is a random variable \(X_{(v \to w)}\). If event \((v \to w)\) occurs, \(X_{(v \to w)}\) has the value \(d(w)\), if not it has the value 0. For each edge \((v, w)\) we have two such variables, the event \(X_{(v \to w)}\) and \(X_{(w \to v)}\). Due to Theorem 5.9, the expected value of the sum \(X\) of all these random variables is at least

\[
\mathbb{E}[X] = \sum_{(v, w) \in E} \mathbb{E}[X_{(v \to w)}] + \mathbb{E}[X_{(w \to v)}] \\
= \sum_{(v, w) \in E} P(\text{Event } (v \to w)) \cdot d(w) + P(\text{Event } (w \to v)) \cdot d(v) \\
\geq \sum_{(v, w) \in E} \frac{d(w)}{d(w) + d(v)} + \frac{d(v)}{d(w) + d(v)} \\
= \sum_{(v, w) \in E} 1 = |E|.
\]

In other words, in expectation all edges are removed in a single phase!?! Probably not. This means that we still counted some edges more than once. Indeed, for an edge \((v, w) \in E\) our random variable \(X\) includes the edge if the event \((v \to w)\) happens, but \(X\) also includes the edge if the event \((x \to u)\) happens. So we may have counted the edge \((v, w)\) twice. Fortunately however, not more than twice, because at most one event \((v \to w)\) and at most one event \((x \to u)\) can happen. If \((u \to w)\) happens, we know that \(v \in N(w)\); hence another \((u \to v)\) cannot happen because \(r(u) = r(w)\). Therefore the random variable \(X\) must be divided by 2. In other words, in expectation at least half of the edges are removed.

**Remarks:**

- This enables us to follow a bound on the expected running time of Algorithm 18 quite easily.

**Theorem 5.11** (Expected running time of Algorithm 18). Algorithm 18 terminates after at most \(3 \log_{4/3} m + 1 \in O(\log n)\) phases in expectation.

Proof: The probability that in a single phase at least a quarter of all edges are removed is at least \(1/3\). For the sake of contradiction, assume not. Then with probability less than \(1/3\) we may be lucky and many (potentially all) edges are removed. With probability more than \(2/3\) less than \(1/4\) of the edges are removed. Hence the expected fraction of removed edges is strictly less than \(1/3 \cdot 1 + 2/3 \cdot 1/4 = 1/2\). This contradicts Lemma 5.10.

Hence, at least every third phase is “good” and removes at least a quarter of the edges. To get rid of all but two edges we need \(\log_{4/3} m\) good phases in expectation. The last two edges will certainly be removed in the next phase. Hence a total of \(3 \log_{4/3} m + 1\) phases are enough in expectation.
5.4. APPLICATIONS

Remarks:

• Sometimes one expects a bit more of an algorithm: Not only should the expected
time to terminate be good, but the algorithm should always terminate quickly. As
this is impossible in randomized algorithms (after all, the random choices may
be “unlucky” all the time!), researchers often settle for a compromise, and just
demand that the probability that the algorithm does not terminate in the specified
time can be made absurdly small. For our algorithm, this can be deduced from
Lemma 5.10 and another standard tool, namely Chernoff’s Bound.

Definition 5.12 (W.h.p.). We say that an algorithm terminates w.h.p. (with high proba-
bility) within $O(t)$ time if it does so with probability at least $1 - 1/n^c$ for any choice of
c $\geq 1$. Here $c$ may affect the constants in the Big-O notation because it is considered
a “tunable constant” and usually kept small.

Definition 5.13 (Chernoff’s Bound). Let $X = \sum_{i=1}^{k} X_i$ be the sum of $k$ independent
0 - 1 random variables. Then Chernoff’s bound states that w.h.p.

$$|X - E[X]| \leq O \left( \log n + \sqrt{E[X] \log n} \right).$$

Corollary 5.14 (Running Time of Algorithm 18). Algorithm 18 terminates w.h.p. in
$O(\log n)$ time.

Proof: In Theorem 5.11 we used that independently of everything that happened before,
in each phase we have a constant probability $p$ that a quarter of the edges are removed.
Call such a phase good. For some constants $C_1$ and $C_2$, let us check after $C_1 \log n +
C_2 \in O(\log n)$ phases, in how many phases at least a quarter of the edges have
been removed. In expectation, these are at least $p(C_1 \log n + C_2)$ many. Now we look at the
random variable $X = \sum_{i=1}^{C_1 \log n + C_2} X_i$, where the $X_i$ are independent 0 - 1 variables
being one with exactly probability $p$. Certainly, if $X$ is at least $x$ with some probability,
then the probability that we have $x$ good phases can only be larger (if no edges are left,
certainly “all” of the remaining edges are removed). To $X$ we can apply Chernoff’s
bound. If $C_1$ and $C_2$ are chosen large enough, they will overcome the constants in
the Big-O from Chernoff’s bound, i.e., w.h.p. it holds that $|X - E[X]| \leq E[X]/2$, implying
$X \geq E[X]/2$. Choosing $C_1$ large enough, we will have w.h.p. sufficiently
many good phases, i.e., the algorithm terminates w.h.p. in $O(\log n)$ phases.

Remarks:

• The algorithm can be improved a bit more even. Drawing random real numbers
in each phase for instance is not necessary. One can achieve the same by sending
only a total of $O(\log n)$ random (and as many non-random) bits over each edge.

• One of the main open problems in distributed computing is whether one can beat
this logarithmic time, or at least achieve it with a deterministic algorithm.

• Let’s turn our attention to applications of MIS next.

5.4 Applications

Definition 5.15 (Matching). Given a graph $G = (V, E)$ a matching is a subset of edges
$M \subseteq E$, such that no two edges in $M$ are adjacent (i.e., where no node is adjacent to
two edges in the matching). A matching is maximal if no edge can be added without violating the above constraint. A matching of maximum cardinality is called maximum. A matching is called perfect if each node is adjacent to an edge in the matching.

Remarks:

- In contrast to MaxIS, a maximum matching can be found in polynomial time (Blossom algorithm by Jack Edmonds), and is also easy to approximate (in fact, already any maximal matching is a 2-approximation).

- An independent set algorithm is also a matching algorithm: Let $G = (V, E)$ be the graph for which we want to construct the matching. The auxiliary graph $G'$ is defined as follows: for every edge in $G$ there is a node in $G'$; two nodes in $G'$ are connected by an edge if their respective edges in $G$ are adjacent. A (maximal) independent set in $G'$ is a (maximal) matching in $G$, and vice versa. Using Algorithm 18 directly produces an $O(\log n)$ bound for maximal matching.

- More importantly, our MIS algorithm can also be used for vertex coloring (Problem 4.1):

Algorithm 19: General Graph Coloring

1. Given a graph $G = (V, E)$ we virtually build a graph $G' = (V', E')$ as follows:
2. Every node $v \in V$ clones itself $d(v) + 1$ times ($v_0, \ldots, v_{d(v)} \in V'$), $d(v)$ being the degree of $v$ in $G$.
3. The edge set $E'$ of $G'$ is as follows:
4. First all clones are in a clique: $(v_i, v_j) \in E'$, for all $v \in V$ and all $0 \leq i < j \leq d(v)$
5. Second all $i$th clones of neighbors in the original graph $G$ are connected: $(u_i, v_i) \in E'$, for all $(u, v) \in E$ and all $0 \leq i \leq \min(d(u), d(v))$.
6. Now we simply run (simulate) the fast MIS Algorithm 18 on $G'$.
7. If node $v_i$ is in the MIS in $G'$, then node $v$ gets color $i$.

Theorem 5.16 (Analysis of Algorithm 19). Algorithm 19 $(\Delta + 1)$-colors an arbitrary graph in $O(\log n)$ time, with high probability, $\Delta$ being the largest degree in the graph.

Proof: Thanks to the clique among the clones at most one clone is in the MIS. And because of the $d(v) + 1$ clones of node $v$ every node will get a free color! The running time remains logarithmic since $G'$ has $O(n^2)$ nodes and the exponent becomes a constant factor when applying the logarithm.

Remarks:

- This solves our open problem from Chapter 4.1!

- Together with Corollary 5.3 we get quite close ties between $(\Delta + 1)$-coloring and the MIS problem.

- However, in general Algorithm 19 is not the best distributed algorithm for $O(\Delta)$-coloring. For fast distributed vertex coloring please check Kothapalli, Onus, Scheideler, Schindelhauer, IPDPS 2006. This algorithm is based on a $O(\log \log n)$ time edge coloring algorithm by Grable and Panconesi, 1997.
5.4. APPLICATIONS

- Computing a MIS also solves another graph problem on graphs of bounded independence.

**Definition 5.17** (Bounded Independence). \( G = (V, E) \) is of bounded independence, if each neighborhood contains at most a constant number of independent (i.e., mutually non-adjacent) nodes.

**Definition 5.18** ((Minimum) Dominating Sets). A dominating set is a subset of the nodes such that each node is in the set or adjacent to a node in the set. A minimum dominating set is a dominating set containing the least possible number of nodes.

Remarks:

- In general, finding a dominating set less than factor \( \log n \) larger than an minimum dominating set is NP-hard.
- Any MIS is a dominating set: if a node was not covered, it could join the independent set.
- In general a MIS and a minimum dominating sets have not much in common (think of a star). For graphs of bounded independence, this is different.

**Corollary 5.19.** On graphs of bounded independence, a constant-factor approximation to a minimum dominating set can be found in time \( O(\log n) \) w.h.p.

Proof: Denote by \( M \) a minimum dominating set and by \( I \) a MIS. Since \( M \) is a dominating set, each node from \( I \) is in \( M \) or adjacent to a node in \( M \). Since the graph is of bounded independence, no node in \( M \) is adjacent to more than constantly many nodes from \( I \). Thus, \(|I| \in O(|M|)\). Therefore, we can compute a MIS with Algorithm 18 and output it as the dominating set, which takes \( O(\log n) \) rounds w.h.p.
Chapter 6

Distributed Sorting

“Indeed, I believe that virtually *every* important aspect of programming arises somewhere in the context of sorting [and searching]!”

– Donald E. Knuth, The Art of Computer Programming

In this chapter we study a classic problem in computer science—sorting—from a distributed computing perspective. In contrast to an orthodox single-processor sorting algorithm, no node has access to all data, instead the to-be-sorted values are distributed. Distributed sorting then boils down to:

**Definition 6.1** (Sorting). *We choose a graph with \( n \) nodes \( v_1, \ldots, v_n \). Initially each node stores a value. After applying a sorting algorithm, node \( v_k \) stores the \( k \)th smallest value.*

**Remarks:**

- What if we route all values to the same central node \( v \), let \( v \) sort the values locally, and then route them to the correct destinations?! According to the message passing model studied in the first few chapters this is perfectly legal. With a star topology sorting finishes in \( O(1) \) time!

**Definition 6.2** (Node Contention). *In each step of a synchronous algorithm, each node can only send and receive \( O(1) \) messages containing \( O(1) \) values, no matter how many neighbors the node has.*

**Remarks:**

- Using Definition 6.2 sorting on a star graph takes linear time.

### 6.1 Array & Mesh

To get a better intuitive understanding of distributed sorting, we start with two simple topologies, the array and the mesh. Let us begin with the array:
Algorithm 20 Odd/Even Sort

1: Given an array of \( n \) nodes \((v_1, \ldots, v_n)\), each storing a value (not sorted).
2: \textbf{repeat}
3: \hspace{1em} Compare and exchange the values at nodes \( i \) and \( i + 1 \), \( i \) odd
4: \hspace{1em} Compare and exchange the values at nodes \( i \) and \( i + 1 \), \( i \) even
5: \textbf{until} done

Remarks:

• The compare and exchange primitive in Algorithm 20 is defined as follows: Let the value stored at node \( i \) be \( v_i \). After the compare and exchange node \( i \) stores value \( \min(v_i, v_{i+1}) \) and node \( i + 1 \) stores value \( \max(v_i, v_{i+1}) \).

• How fast is the algorithm, and how can we prove correctness/efficiency?

• The most interesting proof uses the so-called 0-1 Sorting Lemma. It allows us to restrict our attention to an input of 0’s and 1’s only, and works for any “oblivious comparison-exchange” algorithm. (Oblivious means: Whether you exchange two values must only depend on the relative order of the two values, and not on anything else.)

Lemma 6.3 (0-1 Sorting Lemma). If an oblivious comparison-exchange algorithm sorts all inputs of 0’s and 1’s, then it sorts arbitrary inputs.

Proof. We prove the opposite direction (does not sort arbitrary inputs \( \Rightarrow \) does not sort 0’s and 1’s). Assume that there is an input \( x = x_1, \ldots, x_n \) that is not sorted correctly. Then there is a smallest value \( k \) such that the value at node \( v_k \) after running the sorting algorithm is strictly larger than the \( k \)th smallest value \( x(k) \). Define an input \( x^*_i = 0 \Leftrightarrow x_i \leq x(k), x^*_i = 1 \text{ else} \). Whenever the algorithm compares a pair of 1’s or 0’s, it is not important whether it exchanges the values or not, so we may simply assume that it does the same as on the input \( x \). On the other hand, whenever the algorithm exchanges some values \( x^*_i = 0 \) and \( x^*_j = 1 \), this means that \( x_i \leq x(k) < x_j \). Therefore, in this case the respective compare-exchange operation will do the same on both inputs. We conclude that the algorithm will order \( x^* \) the same way as \( x \), i.e., the output with only 0’s and 1’s will also not be correct. \( \square \)

Theorem 6.4. Algorithm 20 sorts correctly in \( n \) steps.

Proof. Thanks to Lemma 6.3 we only need to consider an array with 0’s and 1’s. Let \( j_1 \) be the node with the rightmost (highest index) 1. If \( j_1 \) is odd (even) it will move in the first (second) step. In any case it will move right in every following step until it reaches the rightmost node \( v_n \). Let \( j_k \) be the node with the \( k \)th rightmost 1. We show by induction that \( j_k \) is not “blocked” anymore (constantly moves until it reaches destination!) after step \( k \). We have already anchored the induction at \( k = 1 \). Since \( j_{k-1} \) moves after step \( k - 1 \), \( j_k \) gets a right 0-neighbor for each step after step \( k \). (For matters of presentation we omitted a couple of simple details.) \( \square \)
6.1. ARRAY & MESH

Algorithm 21 Shearsort
1: We are given a mesh with \( m \) rows and \( m \) columns, \( m \) even, \( n = m^2 \).
2: The sorting algorithm operates in phases, and uses the odd/even sort algorithm on rows or columns.
3: repeat
4: In the odd phases, we sort all the rows, in the even phases, we sort all the columns, such that:
5: Columns are sorted such that the small values move up.
6: Odd rows (1, 3, \ldots, \( m-1 \)) are sorted such that small values move left.
7: Even rows (2, 4, \ldots, \( m \)) are sorted such that small values move right.
8: until done

Remarks:
- Linear time is not very exciting, maybe we can do better by using a different topology? Let’s try a mesh (a.k.a. grid) topology first.

Theorem 6.5. Algorithm 21 sorts \( n \) values in \( \sqrt{n} (\log n + 1) \) time in snake-like order.

Proof. Since the algorithm is oblivious, we can use Lemma 6.3. We show that after a row and a column phase, half of the previously unsorted rows will be sorted. More formally, let us call a row with only 0’s (or only 1’s) clean, a row with 0’s and 1’s is dirty. At any stage, the rows of the mesh can be divided into three regions. In the north we have a region of all-0 rows, in the south all-1 rows, in the middle a region of dirty rows. Initially all rows can be dirty. Since neither row nor column sort will touch already clean rows, we can concentrate on the dirty rows.

First we run an odd phase. Then, in the even phase, we run a peculiar column sorter: We group two consecutive dirty rows into pairs. Since odd and even rows are sorted in opposite directions, two consecutive dirty rows look as follows:

\[
\begin{align*}
00000 & \ldots 11111 \\
11111 & \ldots 00000
\end{align*}
\]

Such a pair can be in one of three states. Either we have more 0’s than 1’s, or more 1’s than 0’s, or an equal number of 0’s and 1’s. Column-sorting each pair will give us at least one clean row (and two clean rows if “0|1 = 1|0”). Then move the cleaned rows north/south and we will be left with half the dirty rows.

At first glance it appears that we need such a peculiar column sorter. However, any column sorter sorts the columns in exactly the same way (we are very grateful to have Lemma 6.3!).

All in all we need \( 2 \log m = \log n \) phases to remain only with 1 dirty row in the middle which will be sorted (not cleaned) with the last row-sort.

Remarks:
- There are algorithms that sort in \( 3m + o(m) \) time on an \( m \) by \( m \) mesh (by diving the mesh into smaller blocks). This is asymptotically optimal, since a value might need to move \( 2m \) times.
• Such a $\sqrt{n}$-sorter is cute, but we are more ambitious. There are non-distributed sorting algorithms such as quicksort, heapsort, or mergesort that sort $n$ values in (expected) $O(n \log n)$ time. Using our $n$-fold parallelism effectively we might therefore hope for a distributed sorting algorithm that sorts in time $O(\log n)$!

6.2 Sorting Networks

In this section we construct a graph topology which is carefully manufactured for sorting. This is a deviation from previous chapters where we always had to work with the topology that was given to us. In many application areas (e.g. peer-to-peer networks, communication switches, systolic hardware) it is indeed possible (in fact, crucial!) that an engineer can build the topology best suited for her application.

**Definition 6.6** (Sorting Networks). A comparator is a device with two inputs $x, y$ and two outputs $x', y'$ such that $x' = \min(x, y)$ and $y' = \max(x, y)$. We construct so-called comparison networks that consist of wires that connect comparators (the output port of a comparator is sent to an input port of another comparator). Some wires are not connected to comparator outputs, and some are not connected to comparator inputs. The first are called input wires of the comparison network, the second output wires. Given $n$ values on the input wires, a sorting network ensures that the values are sorted on the output wires. We will also use the term width to indicate the number of wires in the sorting network.

Remarks:

• The odd/even sorter explained in Algorithm 20 can be described as a sorting network.

• Often we will draw all the wires on $n$ horizontal lines ($n$ being the “width” of the network). Comparators are then vertically connecting two of these lines.

• Note that a sorting network is an oblivious comparison-exchange network. Consequently we can apply Lemma 6.3 throughout this section. An example sorting network is depicted in Figure 6.1.

**Definition 6.7** (Depth). The depth of an input wire is 0. The depth of a comparator is the maximum depth of its input wires plus one. The depth of an output wire of a comparator is the depth of the comparator. The depth of a comparison network is the maximum depth (of an output wire).

**Definition 6.8** (Bitonic Sequence). A bitonic sequence is a sequence of numbers that first monotonically increases, and then monotonically decreases, or vice versa.

Remarks:

• $< 1, 4, 6, 8, 3, 2 >$ or $< 5, 3, 2, 1, 4, 8 >$ are bitonic sequences.

• $< 9, 6, 2, 3, 5, 4 >$ or $< 7, 4, 2, 5, 9, 8 >$ are not bitonic.

• Since we restrict ourselves to 0’s and 1’s (Lemma 6.3), bitonic sequences have the form $0^i1^j0^k$ or $1^i0^j1^k$ for $i, j, k \geq 0$. 

6.2. SORTING NETWORKS

Algorithm 22 Half Cleaner

1: A half cleaner is a comparison network of depth 1, where we compare wire $i$ with wire $i + n/2$ for $i = 1, \ldots, n/2$ (we assume $n$ to be even).

Lemma 6.9. Feeding a bitonic sequence into a half cleaner (Algorithm 22), the half cleaner cleans (makes all-0 or all-1) either the upper or the lower half of the $n$ wires. The other half is bitonic.

Proof. Assume that the input is of the form $0^i1^j0^k$ for $i, j, k \geq 0$. If the midpoint falls into the 0’s, the input is already clean/bitonic and will stay so. If the midpoint falls into the 1’s the half cleaner acts as Shearsort with two adjacent rows, exactly as in the proof of Theorem 6.5. The case $1^i0^j1^k$ is symmetric.

Algorithm 23 Bitonic Sequence Sorter

1: A bitonic sequence sorter of width $n$ ($n$ being a power of 2) consists of a half cleaner of width $n$, and then two bitonic sequence sorters of width $n/2$ each.
2: A bitonic sequence sorter of width 1 is empty.

Lemma 6.10. A bitonic sequence sorter (Algorithm 23) of width $n$ sorts bitonic sequences. It has depth $\log n$.

Proof. The proof follows directly from the Algorithm 23 and Lemma 6.9.

Remarks:

- Clearly we want to sort arbitrary and not only bitonic sequences! To do this we need one more concept, merging networks.
Algorithm 24 Merging Network

1: A merging network of width $n$ is a merger of width $n$ followed by two bitonic sequence sorters of width $n/2$. A merger is a depth-one network where we compare wire $i$ with wire $n - i + 1$, for $i = 1, \ldots, n/2$.

Remarks:

• Note that a merging network is a bitonic sequence sorter where we replace the (first) half-cleaner by a merger.

Lemma 6.11. A merging network of width $n$ (Algorithm 24) merges two sorted input sequences of length $n/2$ each into one sorted sequence of length $n$.

Proof. We have two sorted input sequences. Essentially, a merger does to two sorted sequences what a half cleaner does to a bitonic sequence, since the lower part of the input is reversed. In other words, we can use same argument as in Theorem 6.5 and Lemma 6.9: Again, after the merger step either the upper or the lower half is clean, the other is bitonic. The bitonic sequence sorters complete sorting. \qed

Remarks:

• How do you sort $n$ values when you are able to merge two sorted sequences of size $n/2$? Piece of cake, just apply the merger recursively.

Algorithm 25 Batcher’s “Bitonic” Sorting Network

1: A batcher sorting network of width $n$ consists of two batcher sorting networks of width $n/2$ followed by a merging network of width $n$. (See Figure 6.2.)

2: A batcher sorting network of width 1 is empty.

Figure 6.2: A batcher sorting network
6.3. COUNTING NETWORKS

Theorem 6.12. A sorting network (Algorithm 25) sorts an arbitrary sequence of \(n\) values. It has depth \(O(\log^2 n)\).

Proof. Correctness is immediate: at recursive stage \(k (k = 1, 2, 3, \ldots, \log n)\) we merge \(2^k\) sorted sequences into \(2^{k-1}\) sorted sequences. The depth \(d(n)\) of the sorting network of level \(n\) is the depth of a sorting network of level \(n/2\) plus the depth \(m(n)\) of a merging network with width \(n\). The depth of a sorter of level 1 is 0 since the sorter is empty. Since a merging network of width \(n\) has the same depth as a bitonic sequence sorter of width \(n\), we know by Lemma 6.10 that \(m(n) = \log n\). This gives a recursive formula for \(d(n)\) which solves to \(d(n) = \frac{1}{\log 2} \log^2 n + \frac{1}{\log 2} \log n\).

Remarks:

- Simulating Batcher’s sorting network on an ordinary sequential computer takes time \(O(n \log^2 n)\). As said, there are sequential sorting algorithms that sort in asymptotically optimal time \(O(n \log n)\). So a natural question is whether there is a sorting network with depth \(O(\log n)\). Such a network would have some remarkable advantages over sequential asymptotically optimal sorting algorithms such as heapsort. Apart from being highly parallel, it would be completely oblivious, and as such perfectly suited for a fast hardware solution. In 1983, Ajtai, Komlos, and Szemeredi presented a celebrated \(O(\log n)\) depth sorting network. (Unlike Batcher’s sorting network the constant hidden in the big-O of the “AKS” sorting network is too large to be practical, however.)

- It can be shown that Batcher’s sorting network and similarly others can be simulated by a Butterfly network and other hypercubic networks, see next chapter.

- What if a sorting network is asynchronous?!? Clearly, using a synchronizer we can still sort, but it is also possible to use it for something else. Check out the next section!

6.3 Counting Networks

In this section we address distributed counting, a distributed service which can for instance be used for load balancing.

Definition 6.13 (Distributed Counting). A distributed counter is a variable that is common to all processors in a system and that supports an atomic test-and-increment operation. The operation delivers the system’s counter value to the requesting processor and increments it.

Remarks:

- A naive distributed counter stores the system’s counter value with a distinguished central node. When other nodes initiate the test-and-increment operation, they send a request message to the central node and in turn receive a reply message with the current counter value. However, with a large number of nodes operating on the distributed counter, the central processor will become a bottleneck. There will be a congestion of request messages at the central processor, in other words, the system will not scale.
• Is a scalable implementation (without any kind of bottleneck) of such a distributed counter possible, or is distributed counting a problem which is inherently centralized?!

• Distributed counting could for instance be used to implement a load balancing infrastructure, i.e. by sending the job with counter value \( i \) (modulo \( n \)) to server \( i \) (out of \( n \) possible servers).

**Definition 6.14 (Balancer).** A balancer is an asynchronous flip-flop which forwards messages that arrive on the left side to the wires on the right, the first to the upper, the second to the lower, the third to the upper, and so on.

**Algorithm 26 Bitonic Counting Network.**

1. Take Batcher’s bitonic sorting network of width \( w \) and replace all the comparators with balancers.
2. When a node wants to count, it sends a message to an arbitrary input wire.
3. The message is then routed through the network, following the rules of the asynchronous balancers.
4. Each output wire is completed with a “mini-counter.”
5. The mini-counter of wire \( k \) replies the value “\( k + i \cdot w \)” to the initiator of the \( i \)th message it receives.

**Definition 6.15 (Step Property).** A sequence \( y_0, y_1, \ldots, y_{w-1} \) is said to have the step property, if \( 0 \leq y_i - y_j \leq 1 \), for any \( i < j \).

**Remarks:**

• If the output wires have the step property, then with \( r \) requests, exactly the values \( 1, \ldots, r \) will be assigned by the mini-counters. All we need to show is that the counting network has the step property. For that we need some additional facts...

**Facts 6.16.** For a balancer, we denote the number of consumed messages on the \( i \)th input wire with \( x_i \), \( i = 0, 1 \). Similarly, we denote the number of sent messages on the \( i \)th output wire with \( y_i \), \( i = 0, 1 \). A balancer has these properties:

1. A balancer does not generate output-messages; that is, \( x_0 + x_1 \geq y_0 + y_1 \) in any state.
2. Every incoming message is eventually forwarded. In other words, if we are in a quiescent state (no message in transit), then \( x_0 + x_1 = y_0 + y_1 \).
3. The number of messages sent to the upper output wire is at most one higher than the number of messages sent to the lower output wire: in any state \( y_0 = \lceil (y_0 + y_1)/2 \rceil \) (thus \( y_1 = \lfloor (y_0 + y_1)/2 \rfloor \)).

**Facts 6.17.** If a sequence \( y_0, y_1, \ldots, y_{w-1} \) has the step property,

1. then all its subsequences have the step property,
2. then its even and odd subsequences satisfy
\[
\sum_{i=0}^{w/2-1} y_{2i} = \left\lceil \frac{1}{2} \sum_{i=0}^{w-1} y_i \right\rceil \quad \text{and} \quad \sum_{i=0}^{w/2-1} y_{2i+1} = \left\lfloor \frac{1}{2} \sum_{i=0}^{w-1} y_i \right\rfloor.
\]
Facts 6.18. If two sequences \( x_0, x_1, \ldots, x_{w-1} \) and \( y_0, y_1, \ldots, y_{w-1} \) have the step property,

1. \( \sum_{i=0}^{w-1} x_i = \sum_{i=0}^{w-1} y_i \), then \( x_i = y_i \) for \( i = 0, \ldots, w-1 \).

2. \( \sum_{i=0}^{w-1} x_i = \sum_{i=0}^{w-1} y_i + 1 \), then there exists a unique \( j \) (\( j = 0, 1, \ldots, w-1 \)) such that \( x_j = y_j + 1 \), and \( x_i = y_i \) for \( i = 0, \ldots, w-1, i \neq j \).

Remarks:

- An alternative representation of Batcher’s network has been introduced by Aspnes et al. It is isomorphic to Batcher’s network, and relies on a Merger Network \( M[w] \) which is defined inductively: \( M[w] \) consists of two \( M[w/2] \) (an upper and a lower one) whose output is fed to \( w/2 \) balancers. The upper balancer merges the even subsequence \( x_0, x_2, \ldots, x_{w-2} \), while the lower balancer merges the odd subsequence \( x_1, x_3, \ldots, x_{w-1} \). Call the outputs of these two \( M[w/2] \) subnetworks have the step property, so does the output of \( M[w] \).

- If \( w > 2 \), let \( z \), resp. \( z' \), be the output of the upper respectively lower \( M[w/2] \) subnetwork. Since \( x_0, x_1, \ldots, x_{w/2-1} \) and \( x_{w/2}, x_{w/2+1}, \ldots, x_{w-1} \) both have the step property by assumption, their even and odd subsequences also have the step property (Fact 6.17.1). By induction hypothesis, the output of both \( M[w/2] \) subnetworks have the step property. Let \( Z := \sum_{i=0}^{w/2-1} z_i \) and \( Z' := \sum_{i=0}^{w/2-1} z'_i \). From Fact 6.17.2 we conclude that \( Z = \left( \frac{1}{2} \sum_{i=0}^{w/2-1} x_i \right) + \left( \frac{1}{2} \sum_{i=w/2}^{w-1} x_i \right) \) and \( Z' = \left( \frac{1}{2} \sum_{i=0}^{w/2-1} x_i \right) + \left[ \frac{1}{2} \sum_{i=w/2}^{w-1} x_i \right] \). Since \( [a] + [b] \) and \( [a] + [b] \) differ by at most 1 we know that \( Z \) and \( Z' \) differ by at most 1.

- If \( Z = Z' \), Fact 6.18.1 implies that \( z_i = z'_i \) for \( i = 0, \ldots, w/2-1 \). Therefore, the output of \( M[w] \) is \( y_i = z_{i/2} \) for \( i = 0, \ldots, w-1 \). Since \( z_0, \ldots, z_{w/2-1} \) has the step property, so does the output of \( M[w] \) and the lemma follows.

- If \( Z \) and \( Z' \) differ by 1, Fact 6.18.2 implies that \( z_i = z'_i \) for \( i = 0, \ldots, w/2-1 \), except a unique \( j \) such that \( z_j \) and \( z'_j \) differ by only 1, for \( j = 0, \ldots, w/2-1 \). Let \( l := \min(z_j, z'_j) \). Then, the output \( y_i \) (with \( i < 2j \)) is \( l + 1 \). The output \( y_i \) (with \( i > 2j + 1 \)) is \( l \). The output \( y_{2j} \) and \( y_{2j+1} \) are balanced by the final balancer resulting in \( y_{2j} = l + 1 \) and \( y_{2j+1} = l \). Therefore \( M[w] \) preserves the step property.

A bitonic counting network is constructed to fulfill Lemma 6.19, i.e., the final output comes from a Merger whose upper and lower inputs are recursively merged. Therefore, the following theorem follows immediately.

Theorem 6.20 (Correctness). In a quiescent state, the \( w \) output wires of a bitonic counting network of width \( w \) have the step property.
Remarks:

- Is every sorting network also a counting network? No. But surprisingly, the other direction is true!

**Theorem 6.21 (Counting vs. Sorting).** *If a network is a counting network then it is also a sorting network, but not vice versa.*

**Proof.** There are sorting networks that are not counting networks (e.g. odd/even sort, or insertion sort). For the other direction, let \( C \) be a counting network and \( I(C) \) be the isomorphic network, where every balancer is replaced by a comparator. Let \( I(C) \) have an arbitrary input of 0’s and 1’s; that is, some of the input wires have a 0, all others have a 1. There is a message at \( C \)'s \( i^{th} \) input wire if and only if \( I(C) \)'s \( i \) input wire is 0. Since \( C \) is a counting network, all messages are routed to the upper output wires. \( I(C) \) is isomorphic to \( C \), therefore a comparator in \( I(C) \) will receive a 0 on its upper (lower) wire if and only if the corresponding balancer receives a message on its upper (lower) wire. Using an inductive argument, the 0’s and 1’s will be routed through \( I(C) \) such that all 0’s exit the network on the upper wires whereas all 1’s exit the network on the lower wires. Applying Lemma 6.3 shows that \( I(C) \) is a sorting network. \( \square \)

Remarks:

- We claimed that the counting network is correct. However, it is only correct in a quiescent state.

**Definition 6.22 (Linearizable).** A system is linearizable if the order of the values assigned reflects the real-time order in which they were requested. More formally, if there is a pair of operations \( o_1, o_2 \), where operation \( o_1 \) terminates before operation \( o_2 \) starts, and the logical order is “\( o_2 \) before \( o_1 \)”, then a distributed system is not linearizable.

**Lemma 6.23 (Linearizability).** The bitonic counting network is not linearizable.

**Proof.** Consider the bitonic counting network with width 4 in Figure 6.3: Assume that two inc operations were initiated and the corresponding messages entered the network on wire 0 and 2 (both in light gray color). After having passed the second resp. the first balancer, these traversing messages “fall asleep”; In other words, both messages take unusually long time before they are received by the next balancer. Since we are in an asynchronous setting, this may be the case.

![Figure 6.3: Linearizability Counter Example.](image-url)
In the meantime, another \textit{inc} operation (medium gray) is initiated and enters the network on the bottom wire. The message leaves the network on wire 2, and the \textit{inc} operation is completed.

Strictly afterwards, another \textit{inc} operation (dark gray) is initiated and enters the network on wire 1. After having passed all balancers, the message will leave the network wire 0. Finally (and not depicted in Figure 6.3), the two light gray messages reach the next balancer and will eventually leave the network on wires 1 resp. 3. Because the dark gray and the medium gray operation do conflict with Definition 6.22, the bitonic counting network is not linearizable.

\textbf{Remarks:}

- Note that the example in Figure 6.3 behaves correctly in the quiescent state: Finally, exactly the values 0, 1, 2, 3 are allotted.

- It was shown that linearizability comes at a high price (the depth grows linearly with the width).
Chapter 7

Synchronization

So far, we have mainly studied synchronous algorithms. Generally, asynchronous algorithms are more difficult to obtain. Also it is substantially harder to reason about asynchronous algorithms than about synchronous ones. For instance, computing a BFS tree (Chapter 3) efficiently requires much more work in an asynchronous system. However, many real systems are not synchronous, and we therefore have to design asynchronous algorithms. In this chapter, we will look at general simulation techniques, called synchronizers, that allow running synchronous algorithms in asynchronous environments.

7.1 Basics

A synchronizer generates sequences of clock pulses at each node of the network satisfying the condition given by the following definition.

**Definition 7.1** (valid clock pulse). We call a clock pulse generated at a node \( v \) valid if it is generated after \( v \) received all the messages of the synchronous algorithm sent to \( v \) by its neighbors in the previous pulses.

Given a mechanism that generates the clock pulses, a synchronous algorithm is turned into an asynchronous algorithm in an obvious way: As soon as the \( i \)th clock pulse is generated at node \( v \), \( v \) performs all the actions (local computations and sending of messages) of round \( i \) of the synchronous algorithm.

**Theorem 7.2.** If all generated clock pulses are valid according to Definition 7.1, the above method provides an asynchronous algorithm that behaves exactly the same way as the given synchronous algorithm.

**Proof.** When the \( i \)th pulse is generated at a node \( v \), \( v \) has sent and received exactly the same messages and performed the same local computations as in the first \( i - 1 \) rounds of the synchronous algorithm.

The main problem when generating the clock pulses at a node \( v \) is that \( v \) cannot know what messages its neighbors are sending to it in a given synchronous round. Because there are no bounds on link delays, \( v \) cannot simply wait “long enough” before generating the next pulse. In order satisfy Definition 7.1, nodes have to send additional messages for the purpose of synchronization. The total complexity of the resulting asynchronous algorithm depends on the overhead introduced by the synchronizer. For
a synchronizer $S$, let $T(S)$ and $M(S)$ be the time and message complexities of $S$ for each generated clock pulse. As we will see, some of the synchronizers need an initialization phase. We denote the time and message complexities of the initialization by $T_{\text{init}}(S)$ and $M_{\text{init}}(S)$, respectively. If $T(A)$ and $M(A)$ are the time and message complexities of the given synchronous algorithm $A$, the total time and message complexities $T_{\text{tot}}$ and $M_{\text{tot}}$ of the resulting asynchronous algorithm then become

$$T_{\text{tot}} = T_{\text{init}}(S) + T(A) \cdot (1 + T(S)) \quad \text{and} \quad M_{\text{tot}} = M_{\text{init}}(S) + M(A) + T(A) \cdot M(S),$$

respectively.

Remarks:

• Because the initialization only needs to be done once for each network, we will mostly be interested in the overheads $T(S)$ and $M(S)$ per round of the synchronous algorithm.

**Definition 7.3 (Safe Node).** A node $v$ is safe with respect to a certain clock pulse if all messages of the synchronous algorithm sent by $v$ in that pulse have already arrived at their destinations.

**Lemma 7.4.** If all neighbors of a node $v$ are safe with respect to the current clock pulse of $v$, the next pulse can be generated for $v$.

**Proof.** If all neighbors of $v$ are safe with respect to a certain pulse, $v$ has received all messages of the given pulse. Node $v$ therefore satisfies the condition of Definition 7.1 for generating a valid next pulse.

Remarks:

• In order to detect safety, we require that all algorithms send acknowledgements for all received messages. As soon as a node $v$ has received an acknowledgement for each message that it has sent in a certain pulse, it knows that it is safe with respect to that pulse. Note that sending acknowledgements does not increase the asymptotic time and message complexities.

### 7.2 The Local Synchronizer $\alpha$

**Algorithm 27 Synchronizer $\alpha$ (at node $v$)**

1. wait until $v$ is safe
2. send SAFE to all neighbors
3. wait until $v$ receives SAFE messages from all neighbors
4. start new pulse

Synchronizer $\alpha$ is very simple. It does not need an initialization. Using acknowledgements, each node eventually detects that it is safe. It then reports this fact directly to all its neighbors. Whenever a node learns that all its neighbors are safe, a new pulse is generated. Algorithm 27 formally describes the synchronizer $\alpha$.

**Theorem 7.5.** The time and message complexities of synchronizer $\alpha$ per synchronous round are

$$T(\alpha) = O(1) \quad \text{and} \quad M(\alpha) = O(m).$$
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Proof. Communication is only between neighbors. As soon as all neighbors of a node \( v \) become safe, \( v \) knows of this fact after one additional time unit. For every clock pulse, synchronizer \( \alpha \) sends at most four additional messages over every edge: Each of the nodes may have to acknowledge a message and reports safety.

Remarks:

- Synchronizer \( \alpha \) was presented in a framework, mostly set up to have a common standard to discuss different synchronizers. Without the framework, synchronizer \( \alpha \) can be explained more easily:
  1. Send message to all neighbors, include round information \( i \) and actual data of round \( i \) (if any).
  2. Wait for message of round \( i \) from all neighbors, and go to next round.
- Although synchronizer \( \alpha \) allows for simple and fast synchronization, it produces awfully many messages. Can we do better? Yes.

7.3 The Global Synchronizer \( \beta \)

Algorithm 28 Synchronizer \( \beta \) (at node \( v \))

1: wait until \( v \) is safe
2: wait until \( v \) receives SAFE messages from all its children in \( T \)
3: if \( v \neq \ell \) then
4: send SAFE message to parent in \( T \)
5: wait until PULSE message received from parent in \( T \)
6: end if
7: send PULSE message to children in \( T \)
8: start new pulse

Synchronizer \( \beta \) needs an initialization that computes a leader node \( \ell \) and a spanning tree \( T \) rooted at \( \ell \). As soon as all nodes are safe, this information is propagated to \( \ell \) by a convergecast. The leader then broadcasts this information to all nodes. The details of synchronizer \( \beta \) are given in Algorithm 28.

Theorem 7.6. The time and message complexities of synchronizer \( \beta \) per synchronous round are

\[
T(\beta) = O(\text{diameter}(T)) \leq O(n) \quad \text{and} \quad M(\beta) = O(n).
\]

The time and message complexities for the initialization are

\[
T_{\text{init}}(\beta) = O(n) \quad \text{and} \quad M_{\text{init}}(\beta) = O(m + n \log n).
\]

Proof. Because the diameter of \( T \) is at most \( n - 1 \), the convergecast and the broadcast together take at most \( 2n - 2 \) time units. Per clock pulse, the synchronizer sends at most \( 2n - 2 \) synchronization messages (one in each direction over each edge of \( T \)).

With an improvement (due to Awerbuch) of the GHS algorithm (Algorithm 8) you saw in Chapter 3, it is possible to construct an MST in time \( O(n) \) with \( O(m + n \log n) \) messages in an asynchronous environment. Once the tree is computed, the tree can be made rooted in time \( O(n) \) with \( O(n) \) messages.

\( \square \)
Remarks:

- We now got a time-efficient synchronizer ($\alpha$) and a message-efficient synchronizer ($\beta$), it is only natural to ask whether we can have the best of both worlds. And, indeed, we can. How is that synchronizer called? Quite obviously: $\gamma$.

### 7.4 The Hybrid Synchronizer $\gamma$

![Figure 7.1: A cluster partition of a network: The dashed cycles specify the clusters, cluster leaders are black, the solid edges are the edges of the intracluster trees, and the bold solid edges are the intercluster edges](image)

Synchronizer $\gamma$ can be seen as a combination of synchronizers $\alpha$ and $\beta$. In the initialization phase, the network is partitioned into clusters of small diameter. In each cluster, a leader node is chosen and a BFS tree rooted at this leader node is computed. These trees are called the *intracluster trees*. Two clusters $C_1$ and $C_2$ are called neighboring if there are nodes $u \in C_1$ and $v \in C_2$ for which $(u, v) \in E$. For every two neighboring clusters, an *intercluster edge* is chosen, which will serve for communication between these clusters. Figure 7.1 illustrates this partitioning into clusters. We will discuss the details of how to construct such a partition in the next section. We say that a cluster is safe if all its nodes are safe.

Synchronizer $\gamma$ works in two phases. In a first phase, synchronizer $\beta$ is applied separately in each cluster by using the intracluster trees. Whenever the leader of a cluster learns that its cluster is safe, it reports this fact to all the nodes in the clusters as well as to the leaders of the neighboring clusters. Now, the nodes of the cluster enter the second phase where they wait until all the neighboring clusters are known to be safe and then generate the next pulse. Hence, we essentially apply synchronizer $\alpha$ between clusters. A detailed description is given by Algorithm 29.

**Theorem 7.7.** Let $m_C$ be the number of intercluster edges and let $k$ be the maximum cluster radius (i.e., the maximum distance of a leaf to its cluster leader). The time and
Algorithm 29 Synchronizer $\gamma$ (at node $v$)

1: wait until $v$ is safe
2: wait until $v$ receives SAFE messages from all children in intracluster tree
3: if $v$ is not cluster leader then
4: send SAFE message to parent in intracluster tree
5: wait until CLUSTERSAFE message received from parent
6: end if
7: send CLUSTERSAFE message to all children in intracluster tree
8: send NEIGHBORSAFE message over all intercluster edges of $v$
9: wait until $v$ receives NEIGHBORSAFE messages from all adjacent intercluster edges and all children in intracluster tree
10: if $v$ is not cluster leader then
11: send NEIGHBORSAFE message to parent in intracluster tree
12: wait until PULSE message received from parent
13: end if
14: send PULSE message to children in intracluster tree
15: start new pulse

message complexities of synchronizer $\gamma$ are

\[ T(\gamma) = O(k) \quad \text{and} \quad M(\gamma) = O(n + m_C). \]

Proof. We ignore acknowledgements, as they do not affect the asymptotic complexities. Let us first look at the number of messages. Over every intracluster tree edge, exactly one SAFE message, one CLUSTERSAFE message, one NEIGHBORSAFE message, and one PULSE message is sent. Further, one NEIGHBORSAFE message is sent over every intercluster edge. Because there are less than \( n \) intracluster tree edges, the total message complexity therefore is at most \( 4n + 2m_C = O(n + m_C) \).

For the time complexity, note that the depth of each intracluster tree is at most \( k \). On each intracluster tree, two convergecasts (the SAFE and NEIGHBORSAFE messages) and two broadcasts (the CLUSTERSAFE and PULSE messages) are performed. The time complexity for this is at most \( 4k \). There is one more time unit needed to send the NEIGHBORSAFE messages over the intercluster edges. The total time complexity therefore is at most \( 4k + 1 = O(k) \). \qed

7.5 Network Partition

We will now look at the initialization phase of synchronizer $\gamma$. Algorithm 30 describes how to construct a partition into clusters that can be used for synchronizer $\gamma$. In Algorithm 30, \( B(v, r) \) denotes the ball of radius $r$ around $v$, i.e., \( B(v, r) = \{ u \in V : d(u, v) \leq r \} \) where $d(u, v)$ is the hop distance between $u$ and $v$. The algorithm has a parameter $\rho > 1$. The clusters are constructed sequentially. Each cluster is started at an arbitrary node that has not been included in a cluster. Then the cluster radius is grown as long as the cluster grows by a factor more than $\rho$.

Remarks:

- The algorithm allows a trade-off between the cluster diameter $k$ (and thus the time complexity) and the number of intercluster edges $m_C$ (and thus the message complexity). We will quantify the possibilities in the next section.
Algorithm 30 Cluster construction

1: while unprocessed nodes do
2: select an arbitrary unprocessed node v;
3: \( r := 0 \);
4: while \( |B(v, r + 1)| > \rho |B(v, r)| \) do
5: \( r := r + 1 \)
6: end while
7: makeCluster(\( B(v, r) \)) \( \quad \) // all nodes in \( B(v, r) \) are now processed
8: end while

- Two very simple partitions would be to make a cluster out of every single node or to make one big cluster that contains the whole graph. We then get synchronizers \( \alpha \) and \( \beta \) as special cases of synchronizer \( \gamma \).

Theorem 7.8. Algorithm 30 computes a partition of the network graph into clusters of radius at most \( \log_\rho n \). The number of intercluster edges is at most \( (\rho - 1) \cdot n \).

Proof. The radius of a cluster is initially 0 and does only grow as long as it grows by a factor larger than \( \rho \). Since there are only \( n \) nodes in the graph, this can happen at most \( \log_\rho n \) times.

To count the number of intercluster edges, observe that an edge can only become an intercluster edge if it connects a node at the boundary of a cluster with a node outside a cluster. Consider a cluster \( C \) of size \( |C| \). We know that \( C = B(v, r) \) for some \( v \in V \) and \( r \geq 0 \). Further, we know that \( |B(v, r + 1)| \leq \rho \cdot |B(v, r)| \). The number of nodes adjacent to cluster \( C \) is therefore at most \( |B(v, r + 1) \setminus B(v, r)| \leq \rho \cdot |C| - |C| \). Because there is only one intercluster edge connecting two clusters by definition, the number of intercluster edges adjacent to \( C \) is at most \( (\rho - 1) \cdot |C| \). Summing over all clusters, we get that the total number of intercluster edges is at most \( (\rho - 1) \cdot n \).

Corollary 7.9. Using \( \rho = 2 \), Algorithm 30 computes a clustering with cluster radius at most \( \log_2 n \) and with at most \( n \) intercluster edges.

Corollary 7.10. Using \( \rho = n^{1/k} \), Algorithm 30 computes a clustering with cluster radius at most \( k \) and at most \( O(n^{1+1/k}) \) intercluster edges.

Remarks:

- Algorithm 30 describes a centralized construction of the partitioning of the graph. For \( \rho \geq 2 \), the clustering can be computed by an asynchronous distributed algorithm in time \( O(n) \) with \( O(m + n \log n) \) (reasonably sized) messages (showing this will be part of the exercises).

- It can be shown that the trade-off between cluster radius and number of intercluster edges of Algorithm 30 is asymptotically optimal. There are graphs for which every clustering into clusters of radius at most \( k \) requires \( n^{1+c/k} \) intercluster edges for some constant \( c \).

The above remarks lead to a complete characterization of the complexity of synchronizer \( \gamma \).
Corollary 7.11. The time and message complexities of synchronizer $\gamma$ per synchronous round are

$$T(\gamma) = O(k) \quad \text{and} \quad M(\gamma) = O(n^{1+1/k}).$$

The time and message complexities for the initialization are

$$T_{\text{init}}(\gamma) = O(n) \quad \text{and} \quad M_{\text{init}}(\gamma) = O(m + n \log n).$$

Remarks:

- The synchronizer idea and the synchronizers discussed in this chapter are due to Baruch Awerbuch.

- In Chapter 3, you have seen that by using flooding, there is a very simple synchronous algorithm to compute a BFS tree in time $O(D)$ with message complexity $O(m)$. If we use synchronizer $\gamma$ to make this algorithm asynchronous, we get an algorithm with time complexity $O(n + D \log n)$ and message complexity $O(m + n \log n + D \cdot n)$ (including initialization).

- The synchronizers $\alpha$, $\beta$, and $\gamma$ achieve global synchronization, i.e. every node generates every clock pulse. The disadvantage of this is that nodes that do not participate in a computation also have to participate in the synchronization. In many computations (e.g. in a BFS construction), many nodes only participate for a few synchronous rounds. An improved synchronizer due to Awerbuch and Peleg can exploit such a scenario and achieves time and message complexity $O(\log^3 n)$ per synchronous round (without initialization).

- It can be shown that if all nodes in the network need to generate all pulses, the trade-off of synchronizer $\gamma$ is asymptotically optimal.

- Partitions of networks into clusters of small diameter and coverings of networks with clusters of small diameters come in many variations and have various applications in distributed computations. In particular, apart from synchronizers, algorithms for routing, the construction of sparse spanning subgraphs, distributed data structures, and even computations of local structures such as a MIS or a dominating set are based on some kind of network partitions or covers.

7.6 Clock Synchronization

“A man with one clock knows what time it is – a man with two is never sure.”

Synchronizers can directly be used to give nodes in an asynchronous network a common notion of time. In wireless networks, for instance, many basic protocols need an accurate time. Sometimes a common time in the whole network is needed, often it is enough to synchronize neighbors. The purpose of the time division multiple access (TDMA) protocol is to use the common wireless channel as efficiently as possible, i.e., interfering nodes should never transmit at the same time (on the same frequency). If we use synchronizer $\beta$ to give the nodes a common notion of time, every single clock cycle costs $D$ time units!

Often, each (wireless) node is equipped with an internal clock. Using this clock, it should be possible to divide time into slots, and make each node send (or listen,
or sleep, respectively) in the appropriate slots according to the media access control (MAC) layer protocol used.

However, as it turns out, synchronizing clocks in a network is not trivial. As nodes’ internal clocks are not perfect, they will run at speeds that are time-dependent. For instance, variations in temperature or supply voltage will affect this clock drift. For standard clocks, the drift is in the order of parts per million, i.e., within a second, it will accumulate to a couple of microseconds. Wireless TDMA protocols account for this by introducing guard times. Whenever a node knows that it is about to receive a message from a neighbor, it powers up its radio a little bit earlier to make sure that it does not miss the message even when clocks are not perfectly synchronized. If nodes are badly synchronized, messages of different slots might collide.

In the clock synchronization problem, we are given a network (graph) with $n$ nodes. The goal for each node is to have a logical clock such that the logical clock values are well synchronized, and close to real time. Each node is equipped with a hardware clock, that ticks more or less in real time, i.e., the time between two pulses is arbitrary between $[1-\epsilon, 1+\epsilon]$, for a constant $\epsilon \ll 1$. Similarly as in our asynchronous model, we assume that messages sent over the edges of the graph have a delivery time between $[0, 1]$. In other words, we have a bounded but variable drift on the hardware clocks and an arbitrary jitter in the delivery times. The goal is to design a message-passing algorithm that ensures that the logical clock skew of adjacent nodes is as small as possible at all times.

**Theorem 7.12.** The global clock skew (the logical clock difference between any two nodes in the graph) is $\Omega(D)$, where $D$ is the diameter of the graph.

**Proof.** For a node $u$, let $t_u$ be the logical time of $u$ and let $(u \to v)$ denote a message sent from $u$ to a node $v$. Let $t(m)$ be the time delay of a message $m$ and let $u$ and $v$ be neighboring nodes. First consider a case where the message delays between $u$ and $v$ are $1/2$. Then all the messages sent by $u$ and $v$ at time $i$ according to the clock of the sender arrive at time $i + 1/2$ according to the clock of the receiver. Then consider the following cases

- $t_u = t_v + 1/2, t(u \to v) = 1, t(v \to u) = 0$
- $t_u = t_v - 1/2, t(u \to v) = 0, t(v \to u) = 1$,

where the message delivery time is always fast for one node and slow for the other and the logical clocks are off by $1/2$. In both scenarios, the messages sent at time $i$ according to the clock of the sender arrive at time $i + 1/2$ according to the logical clock of the receiver. Therefore, for nodes $u$ and $v$, both cases with clock drift seem the same as the case with perfectly synchronized clocks. Furthermore, in a linked list of $D$ nodes, the left- and rightmost nodes $l, r$ cannot distinguish $t_l = t_r + D/2$ from $t_l = t_r - D/2$. \qed

**Remarks:**

- From Theorem 7.12, it directly follows that all the clock synchronization algorithms we studied have a global skew of $\Omega(D)$.

- Many natural algorithms manage to achieve a global clock skew of $O(D)$. 

7.6. CLOCK SYNCHRONIZATION

As both the message jitter and hardware clock drift are bounded by constants, it feels like we should be able to get a constant drift between neighboring nodes. As synchronizer \( \alpha \) pays most attention to the local synchronization, we take a look at a protocol inspired by the synchronizer \( \alpha \). A pseudo-code representation for the clock synchronization protocol \( \alpha \) is given in Algorithm 31.

**Algorithm 31** Clock synchronization \( \alpha \) (at node \( v \))

1: repeat
2:   send logical time \( t_v \) to all neighbors
3: end if
5: until done

**Lemma 7.13.** The clock synchronization protocol \( \alpha \) has a local skew of \( \Omega(n) \).

**Proof.** Let the graph be a linked list of \( D \) nodes. We denote the nodes by \( v_1, v_2, \ldots, v_D \) from left to right and the logical clock of node \( v_i \) by \( t_i \). Apart from the left-most node \( v_1 \) all hardware clocks run with speed 1 (real time). Node \( v_1 \) runs at maximum speed, i.e. the time between two pulses is not 1 but \( 1 - \epsilon \). Assume that initially all message delays are 1. After some time, node \( v_1 \) will start to speed up \( v_2 \), and after some more time \( v_2 \) will speed up \( v_3 \), and so on. At some point of time, we will have a clock skew of 1 between any two neighbors. In particular \( t_1 = t_2 + D - 1 \).

Now we start playing around with the message delays. Let \( t_1 = T \). First we set the delay between the \( v_1 \) and \( v_2 \) to 0. Now node \( v_2 \) immediately adjusts its logical clock to \( T \). After this event (which is instantaneous in our model) we set the delay between \( v_2 \) and \( v_3 \) to 0, which results in \( v_3 \) setting its logical clock to \( T \) as well. We perform this successively to all pairs of nodes until \( v_{D-2} \) and \( v_{D-1} \). Now node \( v_{D-1} \) sets its logical clock to \( T \), which indicates that the difference between the logical clocks of \( v_{D-1} \) and \( v_D \) is \( T - (T - (D - 1)) = D - 1 \).

**Remarks:**

- The introduced examples may seem cooked-up, but examples like this exist in all networks, and for all algorithms. Indeed, it was shown that any natural clock synchronization algorithm must have a bad local skew. In particular, a protocol that averages between all neighbors is even worse than the introduced \( \alpha \) algorithm. This algorithm has a clock skew of \( \Omega(D^2) \) in the linked list, at all times.

- Recently, there was a lot of progress in this area, and it was shown that the local clock skew is \( \Theta(\log D) \), i.e., there is a protocol that achieves this bound, and there proof that no algorithm can be better than this bound!

- Note that these are worst-case bounds. In practice, clock drift and message delays may not be the worst possible, typically the speed of hardware clocks changes at a comparatively slow pace and the message transmission times follow a benign probability distribution. If we assume this, better protocols do exist.
Chapter Notes

The idea behind synchronizers is quite intuitive and as such, synchronizers $\alpha$ and $\beta$ were implicitly used in various asynchronous algorithms [Gal76, Cha79, CL85] before being proposed as separate entities. The general idea of applying synchronizers to run synchronous algorithms in asynchronous networks was first introduced by Awerbuch [Awe85a]. His work also formally introduced the synchronizers $\alpha$ and $\beta$, whereas other constructions were presented in [AP90, PU87]. Naturally, as synchronizers are motivated by practical difficulties with local clocks, there are plenty of real life applications. Studies regarding applications can be found in, e.g., [SM86, Awe85b, LTC89, AP90, PU87]. Synchronizers in the presence of network failures have been discussed in [AP88, HS94].

It has been known for a long time that the global clock skew is $\Theta(D)$ [LL84, ST87]. The problem of synchronizing the clocks of nearby nodes was introduced by Fan and Lynch in [LF04]; they proved a surprising lower bound of $\Omega(\log D / \log \log D)$ for the local skew. The first algorithm providing a non-trivial local skew of $O(\sqrt{D})$ was given in [LW06]. Later, matching upper and lower bounds of $\Theta(\log D)$ were given in [LLW10]. The problem has also been studied in a dynamic setting [KLO09, KLLO10].

Clock synchronization is a well-studied problem in practice, for instance regarding the global clock skew in sensor networks, e.g. [EGE02, GKS03, MKSL04, PSJ04]. One more recent line of work is focussing on the problem of minimizing the local clock skew [BvRW07, SW09, LSW09, FW10, FZTS11].
Bibliography


Chapter 8

All-to-All Communication

In the previous chapters, we have mostly considered communication on a particular graph $G = (V, E)$, where any two nodes $u$ and $v$ can only communicate directly if $\{u, v\} \in E$. This is however not always the best way to model a network. In the Internet, for example, every machine (node) is able to “directly” communicate with every other machine via a series of routers. If every node in a network can communicate directly with all other nodes, many problems can be solved easily. For example, assume we have $n$ servers, each hosting an arbitrary number of (numeric) elements. If all servers are interested in obtaining the maximum of all elements, all servers can simultaneously, i.e., in one communication round, send their local maximum element to all other servers. Once these maxima are received, each server knows the global maximum.

Note that we can again use graph theory to model this all-to-all communication scenario: The communication graph is simply the complete graph $K_n := (V, \binom{V}{2})$. If each node can send its entire local state in a single message, then all problems could be solved in 1 communication round in this model! Since allowing unbounded messages is not realistic in most practical scenarios, we restrict the message size: Assuming that all node identifiers and all other variables in the system (such as the numeric elements in the example above) can be described using $O(\log n)$ bits, each node can only send a message of size $O(\log n)$ bits to all other nodes. In other words, only a constant number of identifiers (and elements) can be packed into a single message. Thus, in this model, the limiting factor is the amount of information that can be transmitted in a fixed amount of time. This is fundamentally different from the model we studied before where nodes are restricted to local information about the network graph.

In this chapter, we study one particular problem in this model, the computation of a minimum spanning tree (MST), i.e., we will again look at the construction of a basic network structure. Let us first review the definition of a minimum spanning tree from Chapter 3. We assume that each edge $e$ is assigned a weight $\omega_e$.

**Definition 8.1 (MST).** Given a weighted graph $G = (V, E, \omega)$. The MST of $G$ is a spanning tree $T$ minimizing $\omega(T)$, where $\omega(H) = \sum_{e \in H} \omega_e$ for any subgraph $H \subseteq G$.

**Remarks:**

- Since we have a complete communication graph, the graph has $\binom{n}{2}$ edges in the beginning.
As in Chapter 3, we assume that no two edges of the graph have the same weight. Recall that assumption ensures that the MST is unique. Recall also that this simplification is not essential as one can always break ties by using the IDs of adjacent vertices.

For simplicity, we assume that we have a synchronous model (as we are only interested in the time complexity, our algorithm can be made asynchronous using synchronizer $\alpha$ at no additional cost (cf. Chapter 7). As usual, in every round, every node can send a (potentially different) message to each of its neighbors. In particular, note that the message delay is 1 for every edge $e$ independent of the weight $\omega_e$. As mentioned before, every message can contain a constant number of node IDs and edge weights (and $O(\log n)$ additional bits).

There is a considerable amount of work on distributed MST construction. Table 8.1 lists the most important results for various network diameters $D$. As we have a complete communication network in our model, we focus only on $D = 1$.

### Upper Bounds

<table>
<thead>
<tr>
<th>Graph Class</th>
<th>Time Complexity</th>
<th>Authors</th>
</tr>
</thead>
<tbody>
<tr>
<td>General Graphs</td>
<td>$O(D + \sqrt{n} \cdot \log^* n)$</td>
<td>Kutten, Peleg</td>
</tr>
<tr>
<td>Diameter 2</td>
<td>$O(\log n)$</td>
<td>Lotker, Patt-Shamir, Peleg</td>
</tr>
<tr>
<td>Diameter 1</td>
<td>$O(\log \log n)$</td>
<td>Lotker, Patt-Shamir, Pavlov, Peleg</td>
</tr>
</tbody>
</table>

### Lower Bounds

<table>
<thead>
<tr>
<th>Graph Class</th>
<th>Time Complexity</th>
<th>Authors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter $\Omega(\log n)$</td>
<td>$\Omega(D + \sqrt{n} / \log^2 n)$</td>
<td>Peleg, Rubinovich</td>
</tr>
<tr>
<td>Diameter 4</td>
<td>$\Omega(n^{1/3} / \sqrt{\log n})$</td>
<td>Lotker, Patt-Shamir, Peleg</td>
</tr>
<tr>
<td>Diameter 3</td>
<td>$\Omega(n^{1/4} / \sqrt{\log n})$</td>
<td>Lotker, Patt-Shamir, Peleg</td>
</tr>
</tbody>
</table>

Table 8.1: Time complexity of distributed MST construction

Remarks:

- Note that for graphs of arbitrary diameter $D$, if there are no bounds on the number of messages sent, on the message size, and on the amount of local computations, there is a straightforward generic algorithm to compute an MST in time $D$: In every round, every node sends its complete state to all its neighbors. After $D$
rounds, every node knows the whole graph and can compute any graph structure locally without any further communication.

- In general, the diameter $D$ is also an obvious lower bound for the time needed to compute an MST. In a weighted ring, e.g., it takes time $D$ to find the heaviest edge. In fact, on the ring, time $D$ is required to compute any spanning tree.

In this chapter, we are not concerned with lower bounds, we want to give an algorithm that computes the MST as quickly as possible instead! We again use the following lemma that is proven in Chapter 3.

**Lemma 8.2.** For a given graph $G$ let $T$ be an MST, and let $T' \subseteq T$ be a subgraph (also known as a fragment) of the MST. Edge $e = (u, v)$ is an outgoing edge of $T'$ if $u \in T'$ and $v \notin T'$ (or vice versa). Let the minimum weight outgoing edge of the fragment $T'$ be the so-called blue edge $b(T')$. Then $T' \cup b(T') \subseteq T$.

Lemma 8.2 leads to a straightforward distributed MST algorithm. We start with an empty graph, i.e., every node is a fragment of the MST. The algorithm consists of phases. In every phase, we add the blue edge $b(T')$ of every existing fragment $T'$ to the MST. Algorithm 32 shows how the described simple MST construction can be carried out in a network of diameter 1.

**Algorithm 32** Simple MST Construction (at node $v$)

1: // all nodes always know all current MST edges and thus all MST fragments
2: while $v$ has neighbor $u$ in different fragment do
3: find lowest-weight edge $e$ between $v$ and a node $u$ in a different fragment
4: send $e$ to all nodes
5: determine blue edges of all fragments
6: add blue edges of all fragments to MST, update fragments
7: end while

**Theorem 8.3.** On a complete graph, Algorithm 32 computes an MST in time $O(\log n)$.

**Proof.** The algorithm is correct because of Lemma 8.2. Every node only needs to send a single message to all its neighbors in every phase (line 4). All other computations can be done locally without sending other messages. In particular, the blue edge of a given fragment is the lightest edge sent by any node of that fragment. Because every node always knows the current MST (and all current fragments), lines 5 and 6 can be performed locally.

In every phase, every fragment connects to at least one other fragment. The minimum fragment size therefore at least doubles in every phase. Thus, the number of phases is at most $\log_2 n$.

**Remarks:**

- Algorithm 32 does essentially the same thing as the GHS algorithm (Algorithm 8) discussed in Chapter 3. Because we now have a complete graph and thus every node can communicate with every other node, things become simpler (and also much faster).
• Algorithm 32 does not make use of the fact that a node can send different messages to different nodes. Making use of this possibility will allow us to significantly reduce the running time of the algorithm.

Our goal is now to improve Algorithm 32. We assume that every node has a unique identifier. By sending its own identifier to all other nodes, every node knows the identifiers of all other nodes after one round. Let \( \ell(F) \) be the node with the smallest identifier in fragment \( F \). We call \( \ell(F) \) the leader of fragment \( F \). In order to improve the running time of Algorithm 32, we need to be able to connect every fragment to more than one other fragment in a single phase. Algorithm 33 shows how the nodes can learn about the \( k = |F| \) lightest outgoing edges of each fragment \( F \) (in constant time!).

**Algorithm 33** Fast MST construction (at node \( v \))

1: // all nodes always know all current MST edges and thus all MST fragments
2: repeat
3: \( F \) := fragment of \( v \);
4: \( v \neq F', \) compute min-weight edge \( e_{F'} \) connecting \( v \) to \( F' \)
5: \( v \neq F', \) send \( e_{F'} \) to \( \ell(F') \)
6: if \( v = \ell(F) \) then
7: \( v' \neq F', \) determine min-weight edge \( e_{F,F'} \) between \( F \) and \( F' \)
8: \( k := |F| \)
9: \( E(F) := k \) lightest edges among \( e_{F,F'} \) for \( F' \neq F \)
10: send each edge in \( E(F) \) to a different node in \( F \)
11: end if
12: send edge received from \( \ell(F) \) to all nodes
13: // the following operations are performed locally by each node
14: \( E' :=\) edges received by other nodes
15: AddEdges\( (E') \)
16: until all nodes are in the same fragment

Given this set \( E' \) of edges, each node can locally decide which edges can safely be added to the constructed tree by calling the subroutine AddEdges (Algorithm 34). Note that the set of received edges \( E' \) in line 14 is the same for all nodes. Since all nodes know all current fragments, all nodes add the same set of edges!

Algorithm 34 uses the lightest outgoing edge that connects two fragments (to a larger super-fragment) as long as it is safe to add this edge, i.e., as long as it is clear that this edge is a blue edge. A (super-)fragment that has outgoing edges in \( E' \) that are surely blue edges is called safe. As we will see, a super-fragment \( F \) is safe if all the original fragments that make up \( F \) are still incident to at least one edge in \( E' \) that has not yet been considered. In order to determine whether all lightest outgoing edges in \( E' \) that are incident to a certain fragment \( F \) have been processed, a counter \( c(F) \) is maintained (see line 2). If an edge incident to two (distinct) fragments \( F_i \) and \( F_j \) is processed, both \( c(F_i) \) and \( c(F_j) \) are decremented by 1 (see Line 8).

An edge connecting two distinct super-fragments \( F' \) and \( F'' \) is added if at least one of the two super-fragments is safe. In this case, the two super-fragments are merged into one (new) super-fragment. The new super-fragment is safe if and only if both original super-fragments are safe and the processed edge \( e \) is not the last edge in \( E' \) incident to any of the two fragments \( F_i \) and \( F_j \) that are incident to \( e \), i.e., both counters \( c(F_i) \) and \( c(F_j) \) are still positive (see line 12).
The considered edge \( e \) may not be added for one of two reasons. It is possible that both \( F' \) and \( F'' \) are not safe. Since a super-fragment cannot become safe again, nothing has to be done in this case. The second reason is that \( F' = F'' \). In this case, this single fragment may become unsafe if \( e \) reduced either \( c(F_i) \) or \( c(F_j) \) to zero (see line 18).

**Algorithm 34** AddEdges\((E')\): Given the set of edges \( E' \), determine which edges are added to the MST.

1. Let \( F_1, \ldots, F_r \) be the initial fragments
2. \( \forall F_i \in \{F_1, \ldots, F_r\}, c(F_i) := \# \text{ incident edges in } E' \)
3. Let \( F_1 := F_1, \ldots, F_r := F_r \) be the initial super-fragments
4. \( \forall F_i \in \{F_1, \ldots, F_r\}, \text{safe}(F_i) := \text{true} \)
5. while \( E' \neq \emptyset \) do
6. \( e := \text{lightest edge in } E' \) between the original fragments \( F_i \) and \( F_j \)
7. \( E' := E' \setminus \{e\} \)
8. \( c(F_i) := c(F_i) - 1, c(F_j) := c(F_j) - 1 \)
9. if \( e \) connects super-fragments \( F' \neq F'' \) and \( \text{safe}(F') \) or \( \text{safe}(F'') \) then  
10. add \( e \) to MST  
11. merge \( F' \) and \( F'' \) into one super-fragment \( F_{\text{new}} \)
12. if \( \text{safe}(F') \) and \( \text{safe}(F'') \) and \( c(F_i) > 0 \) and \( c(F_j) > 0 \) then  
13. \( \text{safe}(F_{\text{new}}) := \text{true} \)
14. else  
15. \( \text{safe}(F_{\text{new}}) := \text{false} \)
16. end if  
17. else if \( F' = F'' \) and \( (c(F_i) = 0 \text{ or } c(F_j) = 0) \) then  
18. \( \text{safe}(F') := \text{false} \)
19. end if
20. end while

**Lemma 8.4.** The algorithm only adds MST edges.

**Proof.** We have to prove that at the time we add an edge \( e \) in line 9 of Algorithm 34, \( e \) is the blue edge of some (super-)fragment. By definition, \( e \) is the lightest edge that has not been considered and that connects two distinct super-fragments \( F' \) and \( F'' \). Since \( e \) is added, we know that either \( \text{safe}(F') \) or \( \text{safe}(F'') \) is true. Without loss of generality, assume that \( F' \) is safe. According to the definition of \( \text{safe} \), this means that from each fragment \( F \) in the super-fragment \( F' \) we know at least the lightest outgoing edge, which implies that we also know the lightest outgoing edge, i.e., the blue edge, of \( F' \). Since \( e \) is the lightest edge that connects any two super-fragments, it must hold that \( e \) is exactly the blue edge of \( F' \). Thus, whenever an edge is added, it is an MST edge.

**Theorem 8.5.** Algorithm 33 computes an MST in time \( O(\log \log n) \).

**Proof.** Let \( \beta_k \) denote the size of the smallest fragment after phase \( k \) of Algorithm 33. We first show that every fragment merges with at least \( \beta_k \) other fragments in each phase. Since the size of each fragment after phase \( k \) is at least \( \beta_k \) by definition, we get that the size of each fragment after phase \( k + 1 \) is at least \( \beta_k (\beta_k + 1) \). Assume that a fragment \( F' \), consisting of at least \( \beta_k \) nodes, does not merge with \( \beta_k \) other fragments in phase \( k + 1 \) for any \( k \geq 0 \). Note that \( F' \) cannot be safe because being safe implies...
that there is at least one edge in $E'$ that has not been considered yet and that is the blue edge of $F$. Hence, the phase cannot be completed in this case. On the other hand, if $F$ is not safe, then at least one of its sub-fragments has used up all its $\beta_k$ edges to other fragments. However, such an edge is either used to merge two fragments or it must have been dropped because the two fragments already belong to the same fragment because another edge connected them (in the same phase). In either case, we get that any fragment, and in particular $F$, must merge with at least $\beta_k$ other fragments.

Given that the minimum fragment size grows (quickly) in each phase and that only edges belonging to the MST are added according to Lemma 8.4, we conclude that the algorithm correctly computes the MST. The fact that

$$\beta_{k+1} \geq \beta_k (\beta_k + 1)$$

implies that $\beta_k \geq 2^{2^{k-1}}$ for any $k \geq 1$. Therefore after $1 + \log_2 \log_2 n$ phases, the minimum fragment size is $n$ and thus all nodes are in the same fragment. 

Remarks:

- It is not known whether the $O(\log \log n)$ time complexity of Algorithm 33 is optimal. In fact, no lower bounds are known for the MST construction on graphs of diameter 1 and 2.
- Algorithm 33 makes use of the fact that it is possible to send different messages to different nodes. If we assume that every node always has to send the same message to all other nodes, Algorithm 32 is the best that is known. Also for this simpler case, no lower bound is known.

Chapter Notes

See [DBL03].
Bibliography


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Chapter 9

Wireless Protocols

Wireless communication was one of the major success stories of the last decades. Today, different wireless standards such as wireless local area networks (WLAN) are omnipresent. In some sense, from a distributed computing viewpoint wireless networks are quite simple, as they cannot form arbitrary network topologies. Simplistic models of wireless networks include geometric graph models such as the so-called unit disk graph. Modern models are more robust: The network graph is restricted, e.g., the total number of neighbors of a node which are not adjacent is likely to be small. This observation is hard to capture with purely geometric models, and motivates more advanced network connectivity models such as bounded growth or bounded independence.

However, on the other hand, wireless communication is also more difficult than standard message passing, as for instance nodes are not able to transmit a different message to each neighbor at the same time. And if two neighbors are transmitting at the same time, they interfere, and a node may not be able to decipher anything.

In this chapter we deal with the distributed computing principles of wireless communication: We make the simplifying assumption that all \( n \) nodes are in the communication range of each other, i.e., the network graph is a clique. Nodes share a synchronous time, in each time slot a node can decide to either transmit or receive (or sleep). However, two or more nodes transmitting in a time slot will cause interference. Transmitting nodes are never aware if there is interference because they cannot simultaneously transmit and receive.

9.1 Basics

The basic communication protocol in wireless networks is the medium access control (MAC) protocol. Unfortunately it is difficult to claim that one MAC protocol is better than another, because it all depends on the parameters, such as the network topology, the channel characteristics, or the traffic pattern. When it comes to the principles of wireless protocols, we usually want to achieve much simpler goals. One basic and important question is the following: How long does it take until one node can transmit successfully, without interference? This question is often called the wireless leader election problem (Chapter 2), with the node transmitting alone being the leader.

Clearly, we can use node IDs to solve leader election, e.g., a node with ID \( i \) transmits in time slot \( i \). However, this may be incredibly slow. There are better deterministic solutions, but by and large the best and simplest algorithms are randomized.
Throughout this chapter, we use a random variable \( X \) to denote the number of nodes transmitting in a given slot.

Algorithm 35 Slotted Aloha

1: Every node \( v \) executes the following code:
2: repeat
3: transmit with probability \( 1/n \)
4: until one node has transmitted alone

Theorem 9.1. Using Algorithm 35 allows one node to transmit alone (become a leader) after expected time \( e \).

Proof. The probability for success, i.e., only one node transmitting is

\[
Pr[X = 1] = n \cdot \frac{1}{n} \cdot \left(1 - \frac{1}{n}\right)^{n-1} \approx \frac{1}{e},
\]

where the last approximation is a result from Theorem 9.23 for sufficiently large \( n \). Hence, if we repeat this process \( e \) times, we can expect one success.

Remarks:

- The origin of the name is the ALOHAnet which was developed at the University of Hawaii.

- How does the leader know that it is the leader? One simple solution is a “distributed acknowledgment”. The nodes just continue Algorithm 35, including the ID of the leader in their transmission. So the leader learns that it is the leader.

- One more problem?! Indeed, node \( v \) which managed to transmit the acknowledgment (alone) is the only remaining node which does not know that the leader knows that it is the leader. We can fix this by having the leader acknowledge \( v \)’s successful acknowledgment.

- One can also imagine an unslotted time model. In this model two messages which overlap partially will interfere and no message is received. As everything in this chapter, Algorithm 35 also works in an unslotted time model, with a factor 2 penalty, i.e., the probability for a successful transmission will drop from \( \frac{1}{e} \) to \( \frac{1}{2e} \). Essentially, each slot is divided into \( t \) small time slots with \( t \to \infty \) and the nodes start a new \( t \)-slot long transmission with probability \( \frac{1}{2nt} \).

9.2 Initialization

Sometimes we want the \( n \) nodes to have the IDs \{1, 2, \ldots, n\}. This process is called initialization. Initialization can for instance be used to allow the nodes to transmit one by one without any interference.
9.2. Initialization

9.2.1 Non-Uniform Initialization

Theorem 9.2. If the nodes know $n$, we can initialize them in $O(n)$ time slots.

Proof. We repeatedly elect a leader using e.g., Algorithm 35. The leader gets the next free number and afterwards leaves the process. We know that this works with probability $1/e$. The expected time to finish is hence $e \cdot n$.

Remarks:

- But this algorithm requires that the nodes know $n$ in order to give them IDs from $1, \ldots, n!$ For a more realistic scenario we need a uniform algorithm, i.e., the nodes do not know $n$.

9.2.2 Uniform Initialization with CD

Definition 9.3 (Collision Detection, CD). Two or more nodes transmitting concurrently is called interference. In a system with collision detection, a receiver can distinguish interference from nobody transmitting. In a system without collision detection, a receiver cannot distinguish the two cases.

Let us first present a high-level idea. The set of nodes is recursively partitioned into two non-empty sets, similarly to a binary tree. This is repeated recursively until a set contains only one node which gets the next free ID. Afterwards, the algorithm continues with the next set.

Algorithm 36 RandomizedSplit$(b)$
1: Every node $v$ executes the following code:
2: repeat
3: if $b_v = b$ then
4:   choose $r$ uniformly at random from $\{0, 1\}$
5:   in the next two time slots:
6:     transmit in slot $r$, and listen in other slot
7:   end if
8: until there was at least 1 transmission in both slots
9: if $b_v = b$ then
10: $b_v := b_v + r$ \{append bit $r$ to bitstring $b_v$\}
11: end if
12: if some node $u$ transmitted alone in slot $r \in \{0, 1\}$ then
13: node $u$ gets ID $m$ \{and becomes passive\}
14: $m := m + 1$
15: else
16: RandomizedSplit$(b + r)$
17: end if
Remarks:

- In line 8 the transmitting nodes need to know if they were the only one transmitting. Since we have enough time, we can do a leader election first and use a similar trick as before to ensure this.
- In line 12 we check separately for $r = 0$ and $r = 1$

Algorithm 37 Initialization with Collision Detection

1: Every node $v$ executes the following code:
2: global variable $m := 0$ {number of already identified nodes}
3: local variable $b_v := ''$ {current bitstring of node $v$, initially empty}
4: RandomizedSplit('')

Theorem 9.4. Algorithm 37 correctly initializes the set of nodes in $O(n)$.

Proof. A successful split is defined as a split in which both subsets are non-empty. We know that there are exactly $n-1$ successful splits because we have a binary tree with $n$ leaves and $n-1$ inner nodes. Let us now calculate the probability for creating two non-empty sets from a set of size $k \geq 2$ as

$$Pr[1 \leq X \leq k-1] = 1 - Pr[X = 0] - Pr[X = k] = 1 - \frac{1}{2^k} - \frac{1}{2^k} \geq \frac{1}{2}. $$

Thus, in expectation we need $O(n)$ splits. 

Remarks:

- What if we do not have collision detection?

9.2.3 Uniform Initialization without CD

Let us assume that we have a special node $\ell$ (leader) and let $S$ denote the set of nodes which want to transmit. We now split every time slot from before into two time slots and use the leader to help us distinguish between silence and noise. In the first slot every node from the set $S$ transmits, in the second slot the nodes in $S \cup \{\ell\}$ transmit. This gives the nodes sufficient information to distinguish the different cases (see Table 9.1).

| $|S| = 0$ | nodes in $S$ transmit | nodes in $S \cup \{\ell\}$ transmit |
|---------|----------------------|-----------------------------------|
| no      | yes                  | yes                               |
| $|S| = 1, S = \{\ell\}$ | yes | yes |
| $|S| = 1, S \neq \{\ell\}$ | yes | no |
| $|S| \geq 2$ | no | no |

Table 9.1: Using a leader to distinguish between noise and silence: no represents noise/silence, yes represents a successful transmission.
9.3. LEADER ELECTION

Remarks:

- As such, Algorithm 37 works also without CD, with only a factor 2 overhead.
- More generally, a leader immediately brings CD to any protocol.
- This protocol has an important real life application, for instance when checking out a shopping cart with items which have RFID tags.
- But how do we determine such a leader? And how long does it take until we are “sure” that we have one? Let us repeat the notion of with high probability.

9.3. Leader Election

9.3.1 With High Probability

Definition 9.5 (With High Probability). Some probabilistic event is said to occur with high probability (w.h.p.), if it happens with a probability \( p \geq 1 - \frac{1}{n^c} \), where \( c \) is a constant. The constant \( c \) may be chosen arbitrarily, but it is considered constant with respect to Big-O notation.

Theorem 9.6. Algorithm 35 elects a leader w.h.p. in \( \mathcal{O}(\log n) \) time slots.

Proof. The probability for not electing a leader after \( c \cdot \log n \) time slots, i.e., \( c \log n \) slots without a successful transmission is

\[
\left( 1 - \frac{1}{n^c} \right)^{e \ln n} = \left( 1 - \frac{1}{e} \right)^{e \cdot \ln n} \leq \frac{1}{e^{\ln n^c}} = \frac{1}{n^c}.
\]

\[ \square \]

Remarks:

- What about uniform algorithms, i.e. the number of nodes \( n \) is not known?

9.3.2 Uniform Leader Election

Algorithm 38 Uniform leader election

1: Every node \( v \) executes the following code:
2: for \( k = 1, 2, 3, \ldots \) do
3:  for \( i = 1 \) to \( ck \) do
4:   transmit with probability \( p := 1/2^k \)
5:  if node \( v \) was the only node which transmitted then
6:    \( v \) becomes the leader
7:    break
8:  end if
9:  end for
10: end for

Theorem 9.7. By using Algorithm 38 it is possible to elect a leader w.h.p. in \( \mathcal{O}(\log^2 n) \) time slots if \( n \) is not known.
Proof. Let us briefly describe the algorithm. The nodes transmit with probability \( p = 2^{-k} \) for \( ck \) time slots for \( k = 1, 2, \ldots \). At first \( p \) will be too high and hence there will be a lot of interference. But after \( \log n \) phases, we have \( k \approx \log n \) and thus the nodes transmit with probability \( \approx \frac{1}{n} \). For simplicity’s sake, let us assume that \( n \) is a power of 2. Using the approach outlined above, we know that after \( \log n \) iterations, we have \( p = \frac{1}{n} \). Theorem 9.6 yields that we can elect a leader w.h.p. in \( O(\log n) \) slots. Since we have to try \( \log n \) estimates until \( k \approx n \), the total runtime is \( O(\log^2 n) \). □

Remarks:

- Note that our proposed algorithm has not used collision detection. Can we solve leader election faster in a uniform setting with collision detection?

9.3.3 Fast Leader Election with CD

**Algorithm 39** Uniform leader election with CD

1: Every node \( v \) executes the following code:
2: repeat
3: transmit with probability \( \frac{1}{2} \)
4: if at least one node transmitted then
5: all nodes that did not transmit quit the protocol
6: end if
7: until one node transmits alone

**Theorem 9.8.** With collision detection we can elect a leader using Algorithm 39 w.h.p. in \( O(\log n) \) time slots.

Proof. The number of active nodes \( k \) is monotonically decreasing and always greater than 1 which yields the correctness. A slot is called successful if at most half the active nodes transmit. We can assume that \( k \geq 2 \) since otherwise we would have already elected a leader. We can calculate the probability that a time slot is successful as

\[
Pr[1 \leq X \leq \left\lfloor \frac{k}{2} \right\rfloor] = \frac{1}{2} - Pr[X = 0] = \frac{1}{2} - \frac{1}{2^k} \geq \frac{1}{4}.
\]

Since the number of active nodes at least halves in every successful time slot, \( \log n \) successful time slots are sufficient to elect a leader. Now let \( Y \) be a random variable which counts the number of successful time slots after \( 8 \cdot c \cdot \log n \) time slots. The expected value is \( E[Y] \geq 8 \cdot c \cdot \log n \cdot \frac{1}{2} \geq 2 \cdot \log n \). Since all those time slots are independent from each other, we can apply a Chernoff bound (see Theorem 9.22) with \( \delta = \frac{1}{2} \) which states

\[
Pr[Y < (1 - \delta)E[Y]] \leq e^{-\frac{\delta^2}{2}E[Y]} = e^{-\frac{1}{8} \cdot 2c \cdot \log n} \leq n^{-\alpha}
\]

for any constant \( \alpha \). □

Remarks:

- Can we be even faster?
9.3. LEADER ELECTION

9.3.4 Even Faster Leader Election with CD

Let us first briefly describe an algorithm for this. In the first phase the nodes transmit
with probability $1/2^0, 1/2^1, 1/2^2, \ldots$ until no node transmits. This yields a first
approximation on the number of nodes. Afterwards, a binary search is performed to
determine an even better approximation of $n$. Finally, the third phase finds a constant
approximation of $n$ using a biased random walk. The algorithm stops in any case as
soon as only one node is transmitting which will become the leader.

Algorithm 40
Fast uniform leader election

1: $i := 1$
2: repeat
3: $i := 2 \cdot i$
4: transmit with probability $1/2^i$
5: until no node transmitted
   {End of Phase 1}
6: $l := 2^{i-2}$
7: $u := 2^i$
8: while $l + 1 < u$ do
9: $j := \lceil \frac{l + u}{2} \rceil$
10: transmit with probability $1/2^j$
11: if no node transmitted then
12: $u := j$
13: else
14: $l := j$
15: end if
16: end while
   {End of Phase 2}
17: $k := u$
18: repeat
19: transmit with probability $1/2^k$
20: if no node transmitted then
21: $k := k - 1$
22: else
23: $k := k + 1$
24: end if
25: until exactly one node transmitted

Lemma 9.9. If $j > \log n + \log \log n$, then $Pr[X > 1] \leq \frac{1}{\log n}$.

Proof. The nodes transmit with probability $1/2^j < 1/2^{\log n + \log \log n} = \frac{1}{n \log n}$. The
expected number of nodes transmitting is $E[X] = \frac{n}{n \log n}$. Using Markov’s inequality
(see Theorem 9.21) yields $Pr[X > 1] \leq Pr[X > E[X] \cdot \log n] \leq \frac{1}{\log n}$.

Lemma 9.10. If $j < \log n - \log \log n$, then $P[X = 0] \leq \frac{1}{n}$.

Proof. The nodes transmit with probability $1/2^j < 1/2^{\log n - \log \log n} = \frac{\log n}{n}$. Hence,
the probability for a silent time slot is $(1 - \frac{\log n}{n})^n = e^{-\log n} = \frac{1}{n}$.

Corollary 9.11. If $i > 2 \log n$, then $Pr[X > 1] \leq \frac{1}{\log n}$.
Proof. This follows from Lemma 9.9 since the deviation in this corollary is even larger.

Corollary 9.12. If \( i \leq \frac{1}{2} \log n \), then \( P[X = 0] \leq \frac{1}{2} \).

Proof. This follows from Lemma 9.10 since the deviation in this corollary is even larger.

Lemma 9.13. Let \( v \) be such that \( 2^{v-1} < n \leq 2^v \), i.e., \( v \approx \log n \). If \( k > v + 2 \), then \( Pr[X > 1] \leq \frac{1}{4} \).

Proof. Markov’s inequality yields
\[
Pr[X > 1] = Pr\left[ X > \frac{9k}{n}E[X] \right] < Pr[X > \frac{9k}{2^n}E[X]] < Pr[X > 4E[X]] < \frac{1}{4}.
\]

Lemma 9.14. If \( k < v - 2 \), then \( Pr[X = 0] \leq \frac{1}{4} \).

Proof. A similar analysis is possible to upper bound the probability that a transmission fails if our estimate is too small. We know that \( k \leq v - 2 \) and thus
\[
Pr[X = 0] = \left( 1 - \frac{1}{2k} \right)^n < e^{-\frac{n}{2k}} < e^{-\frac{2^{v-1}}{2^n}} < e^{-2} < \frac{1}{4}.
\]

Lemma 9.15. If \( v-2 \leq k \leq v+2 \), then the probability that exactly one node transmits is constant.

Proof. The transmission probability is \( p = \frac{1}{2^{v+1} n} = \Theta(1/n) \), and the lemma follows with a slightly adapted version of Theorem 9.1.

Lemma 9.16. With probability \( 1 - \frac{1}{\log n} \) we find a leader in phase 3 in \( O(\log \log n) \) time.

Proof. For any \( k \), because of Lemmas 9.13 and 9.14, the random walk of the third phase is biased towards the good area. One can show that in \( O(\log \log n) \) steps one gets \( \Omega(\log \log n) \) good transmissions. Let \( Y \) denote the number of times exactly one node transmitted. With Lemma 9.15 we obtain \( E[Y] = \Omega(\log \log n) \). Now a direct application of a Chernoff bound (see Theorem 9.22) yields that these transmissions elect a leader with probability \( 1 - \frac{1}{\log n} \).

Theorem 9.17. The Algorithm 40 elects a leader with probability of at least \( 1 - \frac{\log \log n}{\log n} \) in time \( O(\log \log n) \).

Proof. From Corollary 9.11 we know that after \( O(\log \log n) \) time slots, the first phase terminates. Since we perform a binary search on an interval of size \( O(\log n) \), the second phase also takes at most \( O(\log \log n) \) time slots. For the third phase we know that \( O(\log \log n) \) slots are sufficient to elect a leader with probability \( 1 - \frac{1}{\log n} \) by Lemma 9.16. Thus, the total runtime is \( O(\log \log n) \).
9.3. LEADER ELECTION

Now we can combine the results. We know that the error probability for every time slot in the first two phases is at most $\frac{1}{\log n}$. Using a union bound (see Theorem 9.20), we can upper bound the probability that no error occurred by $\frac{\log \log n}{\log n}$. Thus, we know that after phase 2 our estimate is at most $\log \log n$ away from $\log n$ with probability of at least $1 - \frac{\log \log n}{\log n}$. Hence, we can apply Lemma 9.16 and thus successfully elect a leader with probability of at least $1 - \frac{\log \log n}{\log n}$ (again using a union bound) in time $O(\log \log n)$.

Remarks:

• Tightening this analysis a bit more, one can elect a leader with probability $1 - \frac{1}{\log n}$ in time $\log \log n + o(\log \log n)$.

• Can we be even faster?

9.3.5 Lower Bound

Theorem 9.18. Any uniform protocol that elects a leader with probability of at least $1 - \frac{1}{\log n}$ must run for at least $\log \log n$ time slots.

Proof. The probability that exactly one node transmits is

$$Pr[X = 1] = n \cdot p \cdot (1 - p)^{n-1}.$$ 

Consider now a system with only 2 nodes. The probability that exactly one transmits is at most

$$Pr[X = 1] = p \cdot (1 - p) \leq \frac{1}{2}.$$

Thus, after $\log \log n$ time slots the probability that a leader was elected is at most

$$1 - \frac{1}{2} \log \log n = 1 - \frac{1}{\log n}.$$ 

9.3.6 Uniform Asynchronous Wakeup without CD

Until now we have assumed that all nodes start the algorithm in the same time slot. But what happens if this is not the case? How long does it take to elect a leader if we want a uniform and anonymous (nodes do not have an identifier and thus cannot base their decision on it) algorithm?

Theorem 9.19. If nodes wake up in an arbitrary (worst-case) way, any algorithm may take $\Omega(n/\log n)$ time slots until a single node can successfully transmit.

Proof. Nodes must transmit at some point, or they will surely never successfully transmit. With a uniform protocol, every node executes the same code. We focus on the first slot where nodes may transmit. No matter what the protocol is, this happens with probability $p$. Since the protocol is uniform, $p$ must be a constant, independent of $n$.

The adversary wakes up $w = \frac{c}{p} \log n$ nodes in each time slot with some constant $c$. All nodes woken up in the first time slot will transmit with probability $p$. We study the event $E_1$ that exactly one of them transmits in that first time slot. Using the inequality $(1 + t/n)^n \leq e^t$ from Lemma 9.23 we get
\[ Pr[E_1] = w \cdot p \cdot (1 - p)^{w-1} \]
\[ = c \ln n \cdot (1 - p)^{(c \ln n - p)} \]
\[ \leq c \ln n \cdot e^{-c \ln n + p} \]
\[ = c \ln n \cdot n^{-e^p} \]
\[ = n^{-e} \cdot O(\log n) \]
\[ < \frac{1}{n^{e-1}} = \frac{1}{n^e}. \]

In other words, w.h.p. that time slot will not be successful. Since the nodes cannot distinguish noise from silence, the same argument applies to every set of nodes which wakes up. Let \( E_n \) be the event that all \( n/w \) time slots will not be successful. Using the inequality \( 1 - p \leq (1 - p/k)^k \) from Lemma 9.24 we get
\[ Pr[E_\alpha] = (1 - Pr(E_1))^{n/w} > \left( 1 - \frac{1}{n^e} \right)^{\Theta(n/\log n)} > 1 - \frac{1}{n^{e^*}}. \]
In other words, w.h.p. it takes more than \( n/w \) time slots until some node can transmit alone.

\[ \square \]

### 9.4 Useful Formulas

In this chapter we have used several inequalities in our proofs. For simplicity’s sake we list all of them in this section.

**Theorem 9.20.** Boole’s inequality or union bound: For a countable set of events \( E_1, E_2, E_3, \ldots \), we have
\[ Pr[\bigcup_i E_i] \leq \sum_i Pr[E_i]. \]

**Theorem 9.21.** Markov’s inequality: If \( X \) is any random variable and \( a > 0 \), then
\[ Pr[|X| \geq a] \leq \frac{E[X]}{a}. \]

**Theorem 9.22.** Chernoff bound: Let \( Y_1, \ldots, Y_n \) be a independent Bernoulli random variables let \( Y := \sum_i Y_i. \) For any \( 0 \leq \delta \leq 1 \) it holds
\[ Pr[Y < (1 - \delta)E[Y]] \leq e^{-\frac{\delta^2 E[Y]}{2}} \]
and for \( \delta > 0 \)
\[ Pr[Y \geq (1 + \delta) \cdot E[Y]] \leq e^{-\frac{\min(\delta, \delta^2)}{3} E[Y]} \]

**Theorem 9.23.** We have
\[ e^t \left( 1 - \frac{t^2}{n} \right) \leq \left( 1 + \frac{t}{n} \right)^n \leq e^t \]
for all \( n \in \mathbb{N}, |t| \leq n. \) Note that
\[ \lim_{n \to \infty} \left( 1 + \frac{t}{n} \right)^n = e^t. \]
Theorem 9.24. For all $p, k$ such that $0 < p < 1$ and $k \geq 1$ we have

$$1 - p \leq (1 - p/k)^k.$$
Chapter 10

Stabilization

A large branch of research in distributed computing deals with fault-tolerance. Being able to tolerate a considerable fraction of failing or even maliciously behaving (“Byzantine”) nodes while trying to reach consensus (on e.g. the output of a function) among the nodes that work properly is crucial for building reliable systems. However, consensus protocols require that a majority of the nodes remains non-faulty all the time.

Can we design a distributed system that survives transient (short-lived) failures, even if all nodes are temporarily failing? In other words, can we build a distributed system that repairs itself?

10.1 Self-Stabilization

Definition 10.1 (Self-Stabilization). A distributed system is self-stabilizing if, starting from an arbitrary state, it is guaranteed to converge to a legitimate state. If the system is in a legitimate state, it is guaranteed to remain there, provided that no further faults happen. A state is legitimate if the state satisfies the specifications of the distributed system.

Remarks:

- What kind of transient failures can we tolerate? An adversary can crash nodes, or make nodes behave Byzantine. Indeed, temporarily an adversary can do harm in even worse ways, e.g. by corrupting the volatile memory of a node (without the node noticing – not unlike the movie Memento), or by corrupting messages on the fly (without anybody noticing). However, as all failures are transient, eventually all nodes must work correctly again, that is, crashed nodes get resurrected, Byzantine nodes stop being malicious, messages are being delivered reliably, and the memory of the nodes is secure.

- Clearly, the read only memory (ROM) must be taboo at all times for the adversary. No system can repair itself if the program code itself or constants are corrupted. The adversary can only corrupt the variables in the volatile random access memory (RAM).

Definition 10.2 (Time Complexity). The time complexity of a self-stabilizing system is the time that passed after the last (transient) failure until the system has converged to a legitimate state again, staying legitimate.
Remarks:

- Self-stabilization enables a distributed system to recover from a transient fault regardless of its nature. A self-stabilizing system does not have to be initialized as it eventually (after convergence) will behave correctly.

- One of the first self-stabilizing algorithms was Dijkstra’s token ring network. A token ring is an early form of a local area network where nodes are arranged in a ring, communicating by a token. The system is correct if there is exactly one token in the ring. Let’s have a look at a simple solution. Given an oriented ring, we simply call the clockwise neighbor parent \((p)\), and the counterclockwise neighbor child \((c)\). Also, there is a leader node \(v_0\). Every node \(v\) is in a state \(S(v) \in \{0, 1, \ldots, n\}\), perpetually informing its child about its state. The token is implicitly passed on by nodes switching state. Upon noticing a change of the parent state \(S(p)\), node \(v\) executes the following code:

\[
\text{Algorithm 41 Self-stabilizing Token Ring}
\]

1: \(\text{if } v = v_0 \text{ then} \)
2: \(\text{if } S(v) = S(p) \text{ then} \)
3: \(S(v) := S(v) + 1 \pmod{n} \)
4: \(\text{end if} \)
5: \(\text{else} \)
6: \(S(v) := S(p) \)
7: \(\text{end if} \)

Theorem 10.3. Algorithm 41 stabilizes correctly.

Proof: As long as some nodes or edges are faulty, anything can happen. In self-stabilization, we only consider the system after it is correct (at time \(t_0\), however starting in an arbitrary state).

Every node apart from leader \(v_0\) will always attain the state of its parent. It may happen that one node after the other will learn the current state of the leader. In this case the system stabilizes after the leader increases its state at most \(n\) time units after time \(t_0\). It may however be that the leader increases its state even if the system is not stable, e.g. because its parent or parent’s parent accidentally had the same state at time \(t_0\).

The leader will increase its state possibly multiple times without reaching stability, however, at some point the leader will reach state \(s\), a state that no other node had at time \(t_0\). (Since there are \(n\) nodes and \(n\) states, this will eventually happen.) At this point the system must stabilize because the leader cannot push for \(s + 1 \pmod{n}\) until every node (including its parent) has \(s\).

After stabilization, there will always be only one node changing its state, i.e., the system remains in a legitimate state.

Remarks:

- Although one might think the time complexity of the algorithm is quite bad, it is asymptotically optimal.
• It can be a lot of fun designing self-stabilizing algorithms. Let us try to build a system, where the nodes organize themselves as a maximal independent set (MIS, Chapter 5):

Algorithm 42 Self-stabilizing MIS

Require: Node IDs

Every node $v$ executes the following code:

1: do atomically
2: Leave MIS if a neighbor with a larger ID is in the MIS
3: Join MIS if no neighbor with larger ID joins MIS
4: Send (node ID, MIS or not MIS) to all neighbors
5: end do

Remarks:

• Note that the main idea of Algorithm 42 is from Algorithm 16, Chapter 5.

• As long as some nodes are faulty, anything can happen: Faulty nodes may for instance decide to join the MIS, but report to their neighbors that they did not join the MIS. Similarly messages may be corrupted during transport. As soon as the system (nodes, messages) is correct, however, the system will converge to a MIS. (The arguments are the same as in Chapter 5).

• Self-stabilizing algorithms always run in an infinite loop, because transient failures can hit the system at any time. Without the infinite loop, an adversary can always corrupt the solution “after” the algorithm terminated.

• The problem of Algorithm 42 is its time complexity, which may be linear in the number of nodes. This is not very exciting. We need something better! Since Algorithm 42 was just the self-stabilizing variant of the slow MIS Algorithm 16, maybe we can hope to “self-stabilize” some of our fast algorithms from Chapter 5?

• Yes, we can! Indeed there is a general transformation that takes any local algorithm (efficient but not fault-tolerant) and turns it into a self-stabilizing algorithm, keeping the same level of efficiency and efficacy. We present the general transformation below.

Theorem 10.4 (Transformation). We are given a deterministic local algorithm $\mathcal{A}$ that computes a solution of a given problem in $k$ synchronous communication rounds. Using our transformation, we get a self-stabilizing system with time complexity $k$. In other words, if the adversary does not corrupt the system for $k$ time units, the solution is stable. In addition, if the adversary does not corrupt any node or message closer than distance $k$ from a node $u$, node $u$ will be stable.

Proof: In the proof, we present the transformation. First, however, we need to be more formal about the deterministic local algorithm $\mathcal{A}$. In $\mathcal{A}$, each node of the network computes its decision in $k$ phases. In phase $i$, node $u$ computes its local variables according to its local variables and received messages of the earlier phases. Then node $u$ sends its messages of phase $i$ to its neighbors. Finally node $u$ receives the messages of phase $i$ from its neighbors. The set of local variables of node $u$ in phase $i$ is given by

For all neighbors: Send each neighbor

As we have seen, many local algorithms are randomized. This brings two ad-

Using our transformation (also known as “local checking”), designing self-

Because of the adversary, node \(A_t N_i\) in induction with Equations (10.1) and (10.2) it follows that at time \(t\) messages take at most 1 time unit to be received at a destination. Hence, using the

\[ m_{i,v} = f_m(u,v,L_u), \text{ for } i \geq 1. \]  

The self-stabilizing algorithm needs to simulate all the \(k\) phases of the local algo-

\[ L_i^k \]  

\( L_i \). (In the very first phase, node \(u\) initializes its local variables with \(L_i^1\).) The message sent from node \(u\) to node \(v\) in phase \(i\) is denoted by \(m_{i,v}^u\). Since the algorithm \(A\) is determinstic, node \(u\) can compute its local variables \(L_i^k\) and messages \(m_{i,v}^u\) of phase \(i\) from its state of earlier phases, by simply applying functions \(f_L\) and \(f_m\). In particular,

\[ L_i^k = f_L(u,L_i^{k-1},m_{i,v}^{k-1}), \text{ for } i > 1, \text{ and} \]

\[ m_{i,v}^u = f_m(u,v,L_i^k), \text{ for } i \geq 1. \]  

(10.1)  

(10.2)

\(\text{Remarks:}\)

- For all neighbors: Send each neighbor \(v\) a message containing the complete row of messages of algorithm \(A\), that is, send the vector \((m_{i,v}^1,\ldots,m_{i,v}^k)\) to neighbor \(v\). Similarly, if neighbor \(u\) receives such a vector from neighbor \(v\), then neighbor \(u\) replaces neighbor \(v\)’s row in the table of incoming messages by the received vector \((m_{i,v}^1,\ldots,m_{i,v}^k)\).

- Because of the adversary, node \(u\) must constantly recompute its local variables (including the initialization) and outgoing message vectors using Functions (10.1) and (10.2) respectively.

The proof is by induction. Let \(N^i(u)\) be the \(i\)-neighborhood of node \(u\) (that is, all nodes within distance \(i\) of node \(u\)). We assume that the adversary has not corrupted any node in \(N^k(u)\) since time \(t_0\). At time \(t_0\) all nodes in \(N^k(u)\) will check and correct their initialization. Following Equation (10.2), at time \(t_0\) all nodes in \(N^k(u)\) will send the correct message entry for the first round \((m_{i,v}^1)\) to all neighbors. Asynchronous messages take at most 1 time unit to be received at a destination. Hence, using the induction with Equations (10.1) and (10.2) it follows that at time \(t_0 + i\), all nodes in \(N^{k-i}(u)\) have received the correct messages \(m_{i,v}^1,\ldots,m_{i,v}^i\). Consequently, at time \(t_0 + k\) node \(u\) has received all messages of local algorithm \(A\) correctly, and will compute the same result value as in \(A\).

\(\square\)

\(\text{Remarks:}\)

- Using our transformation (also known as “local checking”), designing self-

- As we have seen, many local algorithms are randomized. This brings two ad-

   \(A_t N_i\).
random bits. This can be achieved by storing the sufficiently long string along with the program code in the read only memory (ROM). Alternatively, the algorithm might not store the random bit string in its ROM, but only the seed for a random bit generator. We need this in order to keep the adversary from reshuffling random bits until the bits become “bad”, and the expected (or with high probability) efficacy or efficiency guarantees of the original local algorithm \( A \) cannot be guaranteed anymore.

- Since most local algorithms have only a few communication rounds, and only exchange small messages, the memory overhead of the transformation is usually bearable. In addition, information can often be compressed in a suitable way so that for many algorithms message size will remain polylogarithmic. For example, the information of the fast MIS algorithm (Algorithm 18) consists of a series of random values (one for each round), plus two boolean values per round. These boolean values represent whether the node joins the MIS, or whether a neighbor of the node joins the MIS. The order of the values tells in which round a decision is made. Indeed, the series of random bits can even be compressed just into the random seed value, and the neighbors can compute the random values of each round themselves.

- There is hope that our transformation as well gives good algorithms for mobile networks, that is for networks where the topology of the network may change. Indeed, for deterministic local approximation algorithms, this is true: If the adversary does not change the topology of a node’s k-neighborhood in time \( k \), the solution will locally be stable again.

- For randomized local approximation algorithms however, this is not that simple. Assume for example, that we have a randomized local algorithm for the dominating set problem. An adversary can constantly switch the topology of the network, until it finds a topology for which the random bits (which are not really random because these random bits are in ROM) give a solution with a bad approximation ratio. By defining a weaker adversarial model, we can fix this problem. Essentially, the adversary needs to be oblivious, in the sense that it cannot see the solution. Then it will not be possible for the adversary to restart the random computation if the solution is “too good”.

- Self-stabilization is the original approach, and self-organization may be the general theme, but new buzzwords pop up every now and then, e.g. self-configuration, self-management, self-regulation, self-repairing, self-healing, self-optimization, self-adaptivity, or self-protection. Generally all these are summarized as “self-*”. One computing giant coined the term “autonomic computing” to reflect the trend of self-managing distributed systems.

### 10.2 Advanced Stabilization

We finish the chapter with a non-trivial example beyond self-stabilization, showing the beauty and potential of the area: In a small town, every evening each citizen calls all his (or her) friends, asking them whether they will vote for the Democratic or the Republican party at the next election.\(^1\) In our town citizens listen to their friends, and

\(^1\)We are in the US, and as we know from The Simpsons, you “throw your vote away” if you vote for somebody else. As a consequence our example has two parties only.
everybody re-chooses his or her affiliation according to the majority of friends. \(^2\) Is this process going to “stabilize” (in one way or another)?

**Remarks:**

- Is eventually everybody voting for the same party? No.
- Will each citizen eventually stay with the same party? No.
- Will citizens that stayed with the same party for some time, stay with that party forever? No.
- And if their friends also constantly root for the same party? No.
- Will this beast stabilize at all?!? Yes!

**Theorem 10.5 (Dems & Reps).** Eventually every citizen is rooting for the same party every other day.

Proof: To prove that the opinions eventually become fixed or cycle every other day, think of each friendship between citizens as a pair of (directed) edges, one in each direction. Let us say an edge is currently “bad” if the party of the advising friend differs from the next-day’s party of the advised friend. In other words, the edge is bad if the advised friend did not follow the advisor’s opinion (which means that the advisor was in the minority). An edge that is not bad, is “good”.

Consider the out-edges of citizen \(c\) on day \(t\), during which (say) \(c\) roots for the Democrats. Assume that during day \(t\), \(g\) out-edges of \(c\) are good, and \(b\) out-edges are bad. Note that \(g + b\) is the degree of \(c\). Since \(g\) out-edges were good, \(g\) friends of \(c\) root for the Democrats on day \(t + 1\). Likewise, \(b\) friends of \(c\) root for the Republicans on day \(t + 1\). In other words, on the evening of day \(t + 1\) citizen \(c\) will receive \(g\) recommendations for Democrats, and \(b\) for Republicans. We distinguish two cases:

- \(g > b\): In this case, citizen \(c\) will still (or again) root for the Democrats on day \(t + 2\). Note that in this case, on day \(t + 1\), exactly \(g\) in-edges of \(c\) are good, and exactly \(b\) in-edges are bad. In other words, the number of bad out-edges on day \(t\) is exactly the number of bad in-edges on day \(t + 1\).

- \(g < b\): In this case, citizen \(c\) will root for the Republicans on day \(t + 2\). Note that in this case, on day \(t + 1\), exactly \(b\) in-edges of \(c\) are good, and exactly \(g\) in-edges are bad. In other words, the number of bad out-edges on day \(t\) was exactly the number of good in-edges on day \(t + 1\) (and vice versa). Since citizen \(c\) is rooting for the Republicans, the number of bad out-edges on day \(t\) was strictly larger than the number of bad in-edges on day \(t + 1\).

We account for every edge as out-edge on day \(t\), and as in-edge on day \(t + 1\). Since in both of the above cases the number of bad edges does not increase, the total number of bad edges \(B\) cannot increase. In fact, if any node switches its party from day \(t\) to \(t + 2\), we know that the total number of bad edges strictly decreases. But \(B\) cannot decrease forever. Once \(B\) hits its minimum, the system stabilizes in the sense that every citizen will either stick with his or her party forever or flip-flop every day – the system “stabilizes”. \(\square\)

\(^2\)Assume for the sake of simplicity that everybody has an odd number of friends.
Remarks:

- The model can be generalized considerably by, for example, adding weights to vertices (meaning some citizens’ opinions are more important than others), allowing loops (citizens who consider their own current opinions as well), allowing tie-breaking mechanisms, and even allowing different thresholds for party changes.

- How long does it take until the system stabilizes?

- Some of you may be reminded of Conway’s Game of Life: We are given an infinite two-dimensional grid of cells, each of which is in one of two possible states, dead or alive. Every cell interacts with its eight neighbors. In each round, the following transitions occur: Any live cell with fewer than two live neighbors dies, as if caused by loneliness. Any live cell with more than three live neighbors dies, as if by overcrowding. Any live cell with two or three live neighbors lives on to the next generation. Any dead cell with exactly three live neighbors is “born” and becomes a live cell. The initial pattern constitutes the “seed” of the system. The first generation is created by applying the above rules simultaneously to every cell in the seed, births and deaths happen simultaneously, and the discrete moment at which this happens is sometimes called a tick. (In other words, each generation is a pure function of the one before.) The rules continue to be applied repeatedly to create further generations. John Conway figured that these rules were enough to generate interesting situations, including “breeders” with create “guns” which in turn create “gliders”. As such Life in some sense answers an old question by John von Neumann, whether there can be a simple machine that can build copies of itself. In fact Life is Turing complete, that is, as powerful as any computer.

Figure 10.1: A “glider gun”…
Figure 10.2: … in action.
Chapter 11

Locality Lower Bounds

In Chapter 4, we looked at distributed algorithms for coloring. In particular, we saw that rings and rooted trees can be colored with 3 colors in $\log^* n + O(1)$ rounds. In this chapter, we will reconsider the distributed coloring problem. We will look at a classic lower bound by Nathan Linial that shows that the result of Chapter 4 is tight: Coloring rings (and rooted trees) indeed requires $\Omega(\log^* n)$ rounds. In particular, we will prove a lower bound for coloring in the following setting:

- We consider deterministic, synchronous algorithms.
- Message size and local computations are unbounded.
- We assume that the network is a directed ring with $n$ nodes.
- Nodes have unique labels (identifiers) from 1 to $n$.

Remarks:

- A generalization of the lower bound to randomized algorithms is possible. Unfortunately, we will however not have time to discuss this.
- Except for restricting to deterministic algorithms, all the conditions above make a lower bound stronger. Any lower bound for synchronous algorithms certainly also holds for asynchronous ones. A lower bound that is true if message size and local computations are not restricted is clearly also valid if we require a bound on the maximal message size or the amount of local computations. Similarly also assuming that the ring is directed and that node labels are from 1 to $n$ (instead of choosing IDs from a more general domain) strengthen the lower bound.
- Instead of directly proving that 3-coloring a ring needs $\Omega(\log^* n)$ rounds, we will prove a slightly more general statement. We will consider deterministic algorithms with time complexity $r$ (for arbitrary $r$) and derive a lower bound on the number of colors that are needed if we want to properly color an $n$-node ring with an $r$-round algorithm. A 3-coloring lower bound can then be derived by taking the smallest $r$ for which an $r$-round algorithm needs 3 or fewer colors.
11.1 Locality

Let us for a moment look at distributed algorithms more generally (i.e., not only at coloring and not only at rings). Assume that initially, all nodes only know their own label (identifier) and potentially some additional input. As information needs at least \( r \) rounds to travel \( r \) hops, after \( r \) rounds, a node \( v \) can only learn about other nodes at distance at most \( r \). If message size and local computations are not restricted, it is in fact not hard to see, that in \( r \) rounds, a node \( v \) can exactly learn all the node labels and inputs up to distance \( r \). As shown by the following lemma, this allows to transform every deterministic \( r \)-round synchronous algorithm into a simple canonical form.

**Lemma 11.1.** If message size and local computations are not bounded, every deterministic, synchronous \( r \)-round algorithm can be transformed into an algorithm of the form given by Algorithm 43 (i.e., it is possible to first communicate for \( r \) rounds and then do all the computations in the end).

**Proof.** Consider some \( r \)-round algorithm \( A \). We want to show that \( A \) can be brought to the canonical form given by Algorithm 43. First, we let the nodes communicate for \( r \) rounds. Assume that in every round, every node sends its complete state to all of its neighbors (remember that there is no restriction on the maximal message size). By induction, after \( r \) rounds, every node knows the initial state of all other nodes at distance at most \( i \). Hence, after \( r \) rounds, a node \( v \) has the combined initial knowledge of all the nodes in its \( r \)-neighborhood. We want to show that this suffices to locally (at node \( v \)) simulate enough of Algorithm \( A \) to compute all the messages that \( v \) receives in the \( r \) communication rounds of a regular execution of Algorithm \( A \).

Concretely, we prove the following statement by induction on \( i \). For all nodes at distance at most \( r - i + 1 \) from \( v \), node \( v \) can compute all messages of the first \( i \) rounds of a regular execution of \( A \). Note that this implies that \( v \) can compute all the messages it receives from its neighbors during all \( r \) rounds. Because \( v \) knows the initial state of all nodes in the \( r \)-neighborhood, \( v \) can clearly compute all messages of the first round (i.e., the statement is true for \( i = 1 \)). Let us now consider the induction step from \( i \) to \( i + 1 \). By the induction hypothesis, \( v \) can compute the messages of the first \( i \) rounds of all nodes in its \((r - i + 1)\)-neighborhood. It can therefore compute all messages that are received by nodes in the \((r - i)\)-neighborhood in the first \( i \) rounds. This is of course exactly what is needed to compute the messages of round \( i + 1 \) of nodes in the \((r - i)\)-neighborhood. \( \square \)
11.1. LOCALITY

Remarks:

- It is straightforward to generalize the canonical form to randomized algorithms: Every node first computes all the random bits it needs throughout the algorithm. The random bits are then part of the initial state of a node.

Definition 11.2 (r-hop view). We call the collection of the initial states of all nodes in the r-neighborhood of a node v, the r-hop view of v.

Remarks:

- Assume that initially, every node knows its degree, its label (identifier) and potentially some additional input. The r-hop view of a node v then includes the complete topology of the r-neighborhood (excluding edges between nodes at distance r) and the labels and additional inputs of all nodes in the r-neighborhood.

Based on the definition of an r-hop view, we can state the following corollary of Lemma 11.1.

Corollary 11.3. A deterministic r-round algorithm A is a function that maps every possible r-hop view to the set of possible outputs.

Proof. By Lemma 11.1, we know that we can transform Algorithm A to the canonical form given by Algorithm 43. After r communication rounds, every node v knows exactly its r-hop view. This information suffices to compute the output of node v. □

Remarks:

- Note that the above corollary implies that two nodes with equal r-hop views have to compute the same output in every r-round algorithm.

- For coloring algorithms, the only input of a node v is its label. The r-hop view of a node therefore is its labeled r-neighborhood.

- Since we only consider rings, r-hop neighborhoods are particularly simple. The labeled r-neighborhood of a node v (and hence its r-hop view) in a directed ring is simply a \((2r + 1)\)-tuple \((\ell_{-r}, \ell_{-r+1}, \ldots, \ell_0, \ldots, \ell_r)\) of distinct node labels where \(\ell_0\) is the label of v. Assume that for \(i > 0\), \(\ell_i\) is the label of the \(i^{th}\) clockwise neighbor of v and \(\ell_{-i}\) is the label of the \(i^{th}\) counterclockwise neighbor of v. A deterministic coloring algorithm for directed rings therefore is a function that maps \((2r + 1)\)-tuples of node labels to colors.

- Consider two r-hop views \(\mathcal{V}_r = (\ell_{-r}, \ldots, \ell_{r})\) and \(\mathcal{V}_r' = (\ell'_{-r}, \ldots, \ell'_{r})\). If \(\ell'_{i} = \ell_{i+1}\) for \(-r \leq i \leq r - 1\) and if \(\ell'_{i} \neq \ell_{i}\) for \(-r \leq i \leq r\), the r-hop view \(\mathcal{V}_r'\) can be the r-hop view of a clockwise neighbor of a node with r-hop view \(\mathcal{V}_r\). Therefore, every algorithm A that computes a valid coloring needs to assign different colors to \(\mathcal{V}_r\) and \(\mathcal{V}_r'\). Otherwise, there is a ring labeling for which A assigns the same color to two adjacent nodes.
11.2 The Neighborhood Graph

We will now make the above observations concerning colorings of rings a bit more formal. Instead of thinking of an \( r \)-round coloring algorithm as a function from all possible \( r \)-hop views to colors, we will use a slightly different perspective. Interestingly, the problem of understanding distributed coloring algorithms can itself be seen as a classical graph coloring problem.

**Definition 11.4 (Neighborhood Graph).** For a given family of network graphs \( \mathcal{G} \), the \( r \)-neighborhood graph \( N_r(\mathcal{G}) \) is defined as follows. The node set of \( N_r(\mathcal{G}) \) is the set of all possible labeled \( r \)-neighborhoods (i.e., all possible \( r \)-hop views). There is an edge between two labeled \( r \)-neighborhoods \( V_r \) and \( V'_r \) if \( V_r \) and \( V'_r \) can be the \( r \)-hop views of two adjacent nodes.

**Lemma 11.5.** For a given family of network graphs \( \mathcal{G} \), there is an \( r \)-round algorithm that colors graphs of \( \mathcal{G} \) with \( c \) colors iff the chromatic number of the neighborhood graph is \( \chi(N_r(\mathcal{G})) \leq c \).

**Proof.** We have seen that a coloring algorithm is a function that maps every possible \( r \)-hop view to a color. Hence, a coloring algorithm assigns a color to every node of the neighborhood graph \( N_r(\mathcal{G}) \). If two \( r \)-hop views \( V_r \) and \( V'_r \) can be the \( r \)-hop views of two adjacent nodes \( u \) and \( v \) (for some labeled graph in \( \mathcal{G} \)), every correct coloring algorithm must assign different colors to \( V_r \) and \( V'_r \). Thus, specifying an \( r \)-round coloring algorithm for a family of network graphs \( \mathcal{G} \) is equivalent to coloring the respective neighborhood graph \( N_r(\mathcal{G}) \).

Remarks:

- If an algorithm is non-uniform, i.e., the nodes know \( n \), we can see this as having different neighborhood graphs for different values of \( n \) (as opposed to a disconnected neighborhood graph).
- This does not make much of a difference for coloring algorithms on the ring, as we are interested in neighborhoods that are much smaller than \( n \).

Instead of directly defining the neighborhood graph for directed rings, we define directed graphs \( B_{k,n} \) that are closely related to the neighborhood graph. Let \( k \) and \( n \) be two positive integers and assume that \( n \geq k \). The node set of \( B_{k,n} \) contains all \( k \)-tuples of increasing node labels (\( [n] = \{1, \ldots, n\} \)):

\[
V[B_{k,n}] = \{ (\alpha_1, \ldots, \alpha_k) : \alpha_i \in [n], i < j \rightarrow \alpha_i < \alpha_j \} \quad (11.1)
\]

For \( \alpha = (\alpha_1, \ldots, \alpha_k) \) and \( \beta = (\beta_1, \ldots, \beta_k) \) there is a directed edge from \( \alpha \) to \( \beta \) iff

\[
\forall i \in \{1, \ldots, k-1\} : \beta_i = \alpha_{i+1}. \quad (11.2)
\]

**Lemma 11.6.** Viewed as an undirected graph, the graph \( B_{2r+1,n} \) is a subgraph of the \( r \)-neighborhood graph of directed \( n \)-node rings with node labels from \( [n] \).

**Proof.** The claim follows directly from the observations regarding \( r \)-hop views of nodes in a directed ring from Section 11.1. The set of \( k \)-tuples of increasing node labels is a subset of the set of \( k \)-tuples of distinct node labels. Two nodes of \( B_{2r+1,n} \) are connected by a directed edge iff the two corresponding \( r \)-hop views are connected by a directed edge in the neighborhood graph. Note that if there is an edge between \( \alpha \) and \( \beta \) in \( B_{k,n} \), \( \alpha_1 \neq \beta_1 \) because the node labels in \( \alpha \) and \( \beta \) are increasing. \( \square \)
11.2. THE NEIGHBORHOOD GRAPH

To determine a lower bound on the number of colors an $r$-round algorithm needs for directed $n$-node rings, it therefore suffices to determine a lower bound on the chromatic number of $B_{2r+1,n}$. To obtain such a lower bound, we need the following definition.

**Definition 11.7** (Diline Graph). The directed line graph (diline graph) $DL(G)$ of a directed graph $G = (V, E)$ is defined as follows. The node set of $DL(G)$ is $V[DL(G)] = E$. There is a directed edge $((w, x), (y, z))$ between $(w, x) \in E$ and $(y, z) \in E$ iff $x = y$, i.e., if the first edge ends where the second one starts.

**Lemma 11.8.** If $n > k$, the graph $B_{k+1,n}$ can be defined recursively as follows:

$$B_{k+1,n} = DL(B_{k,n}).$$

**Proof.** The edges of $B_{k,n}$ are pairs of $k$-tuples $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_k)$ that satisfy Conditions (11.1) and (11.2). Because the last $k - 1$ labels in $\alpha$ are equal to the first $k - 1$ labels in $\beta$, the pair $(\alpha, \beta)$ can be represented by a $(k + 1)$-tuple $\gamma = (\gamma_1, \ldots, \gamma_{k+1})$ with $\gamma_1 = \alpha_1, \gamma_i = \beta_1, \alpha_i$, for $2 \leq i \leq k$, and $\gamma_{k+1} = \beta_k$. Because the labels in $\alpha$ and the labels in $\beta$ are increasing, the labels in $\gamma$ are increasing as well. The two graphs $B_{k+1,n}$ and $DL(B_{k,n})$ therefore have the same node sets. There is an edge between two nodes $(\Omega_1, \beta_1)$ and $(\Omega_2, \beta_2)$ of $DL(B_{k,n})$ if $\beta_1 = \alpha_2$. This is equivalent to requiring that the two corresponding $(k + 1)$-tuples $\gamma_1$ and $\gamma_2$ are neighbors in $B_{k+1,n}$, i.e., that the last $k$ labels of $\gamma_1$ are equal to the first $k$ labels of $\gamma_2$.

The following lemma establishes a useful connection between the chromatic numbers of a directed graph $G$ and its diline graph $DL(G)$.

**Lemma 11.9.** For the chromatic numbers $\chi(G)$ and $\chi(DL(G))$ of a directed graph $G$ and its diline graph, it holds that

$$\chi(DL(G)) \geq \log_2(\chi(G)).$$

**Proof.** Given a $c$-coloring of $DL(G)$, we show how to construct a $2^c$ coloring of $G$. The claim of the lemma then follows because this implies that $\chi(G) \leq 2^{\chi(DL(G))}$.

Assume that we are given a $c$-coloring of $DL(G)$. A $c$-coloring of the diline graph $DL(G)$ can be seen as a coloring of the edges of $G$ such that no two adjacent edges have the same color. For a node $v$ of $G$, let $S_v$ be the set of colors of its outgoing edges. Let $u$ and $v$ be two nodes such that $G$ contains a directed edge $(u, v)$ from $u$ to $v$ and let $x$ be the color of $(u, v)$. Clearly, $x \in S_v$ because $(u, v)$ is an outgoing edge of $u$. Because adjacent edges have different colors, no outgoing edge $(v, w)$ of $v$ can have color $x$. Therefore $x \not\in S_u$. This implies that $S_u \not= S_v$. We can therefore use these color sets to obtain a vertex coloring of $G$, i.e., the color of $u$ is $S_u$ and the color of $v$ is $S_v$. Because the number of possible subsets of $[c]$ is $2^c$, this yields a $2^c$-coloring of $G$.

Let $\log^{(i)}$ be the $i$-fold application of the base-2 logarithm to $x$:

$$\log^{(i)} x = \log_2 x, \quad \log^{(i+1)} x = \log_2(\log^{(i)} x).$$

Remember from Chapter 4 that

$$\log^* x = 1 \text{ if } x \leq 2, \quad \log^* x = 1 + \min\{i : \log^{(i)} x \leq 2\}.$$ 

For the chromatic number of $B_{k,n}$, we obtain
Lemma 11.10. For all \( n \geq 1 \), \( \chi(\mathcal{B}_{1,n}) = n \). Further, for \( n \geq k \geq 2 \), \( \chi(\mathcal{B}_{k,n}) \geq \log^{(k-1)} n \).

Proof. For \( k = 1 \), \( \mathcal{B}_{k,n} \) is the complete graph on \( n \) nodes with a directed edge from node \( i \) to node \( j \) iff \( i < j \). Therefore, \( \chi(\mathcal{B}_{1,n}) = n \). For \( k > 2 \), the claim follows by induction and Lemmas 11.8 and 11.9.

This finally allows us to state a lower bound on the number of rounds needed to color a directed ring with 3 colors.

Theorem 11.11. Every deterministic, distributed algorithm to color a directed ring with 3 or less colors needs at least \( (\log^* n) / 2 - 1 \) rounds.

Proof. Using the connection between \( \mathcal{B}_{k,n} \) and the neighborhood graph for directed rings, it suffices to show that \( \chi(\mathcal{B}_{2r+1,n}) > 3 \) for all \( r < (\log^* n) / 2 - 1 \). From Lemma 11.10, we know that \( \chi(\mathcal{B}_{2r+1,n}) \geq \log^{(2r)} n \). To obtain \( \log^{(2r)} n \leq 2 \), we need \( r \geq (\log^* n) / 2 - 1 \). Because \( \log_2 3 < 2 \), we therefore have \( \log^{(2r)} n > 3 \) if \( r < (\log^* n) / 2 - 1 \).

Corollary 11.12. Every deterministic, distributed algorithm to compute an MIS of a directed ring needs at least \( \log^* n / 2 - O(1) \) rounds.

Remarks:

- It is straightforward to see that also for a constant \( c > 3 \), the number of rounds needed to color a ring with \( c \) or less colors is \( \log^* n / 2 - O(1) \).
- There basically (up to additive constants) is a gap of a factor of 2 between the \( \log^* n + O(1) \) upper bound of Chapter 4 and the \( \log^* n / 2 - O(1) \) lower bound of this chapter. It is possible to show that the lower bound is tight, even for undirected rings (for directed rings, this will be part of the exercises).
- The presented lower bound is due to Nathan Linial. The lower bound is also true for randomized algorithms. The generalization for randomized algorithms was done by Moni Naor.
- Alternatively, the lower bound can also be presented as an application of Ramsey’s theory. Ramsey’s theory is best introduced with an example: Assume you host a party, and you want to invite people such that there are no three people who mutually know each other, and no three people which are mutual strangers. How many people can you invite? This is an example of Ramsey’s theorem, which says that for any given integer \( c \), and any given integers \( n_1, \ldots, n_c \), there is a Ramsey number \( R(n_1, \ldots, n_c) \), such that if the edges of a complete graph with \( R(n_1, \ldots, n_c) \) nodes are colored with \( c \) different colors, then for some color \( i \) the graph contains some complete subgraph of color \( i \) of size \( n_i \). The special case in the party example is looking for \( R(3,3) \).
- Ramsey theory is more general, as it deals with hyperedges. A normal edge is essentially a subset of two nodes; a hyperedge is a subset of \( k \) nodes. The party example can be explained in this context: We have (hyper)edges of the form \( \{i,j\} \), with \( 1 \leq i, j \leq n \). Choosing \( n \) sufficiently large, coloring the edges with two colors must exhibit a set \( S \) of 3 edges \( \{i,j\} \subseteq \{v_1, v_2, v_3\} \), such that all edges in \( S \) have the same color. To prove our coloring lower bound using
Ramsey theory, we form all hyperedges of size $k = 2r + 1$, and color them with 3 colors. Choosing $n$ sufficiently large, there must be a set $S = \{v_1, \ldots, v_{k+1}\}$ of $k + 1$ identifiers, such that all $k + 1$ hyperedges consisting of $k$ nodes from $S$ have the same color. Note that both $\{v_1, \ldots, v_k\}$ and $\{v_2, \ldots, v_{k+1}\}$ are in the set $S$, hence there will be two neighboring views with the same color. Ramsey theory shows that in this case $n$ will grow as a power tower (tetration) in $k$. Thus, if $n$ is so large that $k$ is smaller than some function growing like $\log^* n$, the coloring algorithm cannot be correct.

- The neighborhood graph concept can be used more generally to study distributed graph coloring. It can for instance be used to show that with a single round (every node sends its identifier to all neighbors) it is possible to color a graph with $(1 + o(1))\Delta^2 \ln n$ colors, and that every one-round algorithm needs at least $\Omega(\Delta^2 / \log^2 \Delta + \log \log n)$ colors.

- One may also extend the proof to other problems, for instance one may show that a constant approximation of the minimum dominating set problem on unit disk graphs costs at least $\log^* n$ time.

- Using $r$-hop views and the fact that nodes with equal $r$-hop views have to make the same decisions is the basic principle behind almost all locality lower bounds (in fact, we are not aware of a locality lower bound that does not use this principle). Using this basic technique (but a completely different proof otherwise), it is for instance possible to show that computing an MIS (and many other problems) in a general graph requires at least $\Omega(\sqrt{\log n / \log \log n})$ and $\Omega(\log \Delta / \log \log \Delta)$ rounds.
Chapter 12

Social Networks

Distributed computing is applicable in various contexts. This lecture exemplarily studies one of these contexts, social networks, an area of study whose origins date back a century. To give you a first impression, consider Figure 12.1.

Figure 12.1: This graph shows the social relations between the members of a karate club, studied by anthropologist Wayne Zachary in the 1970s. Two people (nodes) stand out, the instructor and the administrator of the club, both happen to have many friends among club members. At some point, a dispute caused the club to split into two. Can you predict how the club partitioned? (If not, just search the Internet for Zachary and Karate.)
12.1 Small-World Networks

Back in 1929, Frigyes Karinthy published a volume of short stories that postulated that the world was “shrinking” because human beings were connected more and more. Some claim that he was inspired by radio network pioneer Guglielmo Marconi’s 1909 Nobel Prize speech. Despite physical distance, the growing density of human “networks” renders the actual social distance smaller and smaller. As a result, it is believed that any two individuals can be connected through at most five (or so) acquaintances, i.e., within six hops.

- The topic was hot in the 1960s. For instance, in 1964, Marshall McLuhan coined the metaphor “Global Village”. He wrote: “As electrically contracted, the globe is no more than a village”. He argues that due to the almost instantaneous reaction times of new (“electric”) technologies, each individual inevitably feels the consequences of his actions and thus automatically deeply participates in the global society. McLuhan understood what we now can directly observe – real and virtual world are moving together. He realized that the transmission medium, rather than the transmitted information is at the core of change, as expressed by his famous phrase “the medium is the message”.

- This idea has been followed ardently in the 1960s by several sociologists, first by Michael Gurevich, later by Stanley Milgram. Milgram wanted to know the average path length between two “random” humans, by using various experiments, generally using randomly chosen individuals from the US Midwest as starting points, and a stockbroker living in a suburb of Boston as target. The starting points were given name, address, occupation, plus some personal information about the target. They were asked to send a letter to the target. However, they were not allowed to directly send the letter, rather, they had to pass it to somebody they knew on first-name basis and that they thought to have a higher probability to know the target person. This process was repeated, until somebody knew the target person, and could deliver the letter. Shortly after starting the experiment, letters have been received. Most letters were lost during the process, but if they arrived, the average path length was about 5.5. The observation that the entire population is connected by short acquaintance chains got later popularized by the terms “six degrees of separation” and “small world”.

- Statisticians tried to explain Milgram’s experiments, by essentially giving network models that allowed for short diameters, i.e., each node is connected to each other node by only a few hops. Until today there is a thriving research community in statistical physics that tries to understand network properties that allow for “small world” effects.

- One of the keywords in this area are power-law graphs, networks were node degrees are distributed according to a power-law distribution, i.e. the number of nodes with degree $\delta$ is proportional to $\delta^{-\alpha}$, for some $\alpha > 1$. Such power-law graphs have been witnessed in many application areas, apart from social networks also in the web, or in Biology or Physics.

- Obviously, two power-law graphs might look and behave completely differently, even if $\alpha$ and the number of edges is exactly the same.
One well-known model towards this end is the Watts-Strogatz model. Watts and Strogatz argued that social networks should be modeled by a combination of two networks: As the basis we take a network that has a large cluster coefficient . . .

**Definition 12.1.** The cluster coefficient of a network is defined by the probability that two friends of a node are likely to be friends as well, summing up over all the nodes.

. . . then we augment such a graph with random links, every node for instance points to a constant number of other nodes, chosen uniformly at random. This augmentation represents acquaintances that connect nodes to parts of the network that would otherwise be far away.

**Remarks:**

- Without further information, knowing the cluster coefficient is of questionable value: Assume we arrange the nodes in a grid. Technically, if we connect each node to its four closest neighbors, the graph has cluster coefficient 0, since there are no triangles; if we instead connect each node with its eight closest neighbors, the cluster coefficient is 3/7. The cluster coefficient is quite different, even though both networks have similar characteristics.

This is interesting, but not enough to really understand what is going on. For Milgram’s experiments to work, it is not sufficient to connect the nodes in a certain way. In addition, the nodes themselves need to know how to forward a message to one of their neighbors, even though they cannot know whether that neighbor is really closer to the target. In other words, nodes are not just following physical laws, but they make decisions themselves. In contrast to those mathematicians that worked on the problem earlier, Jon Kleinberg understood that Milgram’s experiment essentially shows that social networks are “navigable”, and that one can only explain it in terms of a greedy routing.

In particular, Kleinberg set up an artificial network with nodes on a grid topology, plus some additional random links per node. In a quantitative study he showed that the random links need a specific distance distribution to allow for efficient greedy routing. This distribution marks the sweet spot for any navigable network.

**Definition 12.2 (Augmented Grid).** We take $n = m^2$ nodes $(i, j) \in V = \{1, \ldots, m\}^2$ that are identified with the lattice points on an $m \times m$ grid. We define the distance between two nodes $(i, j)$ and $(k, \ell)$ as $d((i, j), (k, \ell)) = |k - i| + |\ell - j|$ as the distance between them on the $m \times m$ lattice. The network is modeled using a parameter $\alpha \geq 0$. Each node $u$ has a directed edge to every lattice neighbor. These are the local contacts of a node. In addition, each node also has an additional random link (the long-range contact). For all $u$ and $v$, the long-range contact of $u$ points to node $v$ with probability proportional to $d(u, v)^{-\alpha}$, i.e., with probability $d(u, v)^{-\alpha} / \sum_{w \in V \setminus\{u\}} d(u, w)^{-\alpha}$. Figure 12.2 illustrates the model.

**Remarks:**

- The network model has the following geographic interpretation: nodes (individuals) live on a grid and know their neighbors on the grid. Further, each node has some additional acquaintances throughout the network.

- The parameter $\alpha$ controls how the additional neighbors are distributed across the grid. If $\alpha = 0$, long-range contacts are chosen uniformly at random (as in the
CHAPTER 12. SOCIAL NETWORKS

Figure 12.2: Augmented grid with $m = 6$

Watts-Strogatz model). As $\alpha$ increases, long-range contacts become shorter on average. In the extreme case, if $\alpha \to \infty$, all long-range contacts are to immediate neighbors on the grid.

- It can be shown that as long as $\alpha \leq 2$, the diameter of the resulting graph is polylogarithmic in $n$ (polynomial in $\log n$) with high probability. In particular, if the long-range contacts are chosen uniformly at random ($\alpha = 0$), the diameter is $O(\log n)$.

Since the augmented grid contains random links, we do not know anything for sure about how the random links are distributed. In theory, all links could point to the same node! However, this is almost certainly not the case. Formally this is captured by the term with high probability.

**Definition 12.3 (With High Probability).** Some probabilistic event is said to occur with high probability (w.h.p.), if it happens with a probability $p \geq 1 - 1/n^c$, where $c$ is a constant. The constant $c$ may be chosen arbitrarily, but it is considered constant with respect to Big-O notation.

**Remarks:**

- For instance, a running time bound of $c \log n$ or $c^{\log n} + 5000c$ with probability at least $1 - 1/n^c$ would be $O(\log n)$ w.h.p., but a running time of $n^c$ would not be $O(n)$ w.h.p. since $c$ might also be 50.
This definition is very powerful, as any polynomial (in $n$) number of statements that hold w.h.p. also holds w.h.p. at the same time, regardless of any dependencies between random variables!

**Theorem 12.4.** The diameter of the augmented grid with $\alpha = 0$ is $O(\log n)$ with high probability.

**Proof Sketch.** For simplicity, we will only show that we can reach a node $w$ starting from some node $v$. However, it can be shown that (essentially) each of the intermediate claims holds with high probability, which then by means of the union bound yields that all of the claims hold simultaneously with high probability for all pairs of nodes.

Let $N_g$ be the $\lfloor \log n \rfloor$-hop neighborhood of $v$ on the grid, containing $\Omega(\log^2 n)$ nodes. Each of the nodes in $N_g$ has a random link, probably leading to distant parts of the graph. As long as we have reached only $o(n)$ nodes, any new random link will with probability $1 - o(1)$ lead to a node for which none of its grid neighbors has been visited yet. Thus, in expectation we find almost $|N_g|$ new nodes whose neighbors are “fresh”. Using their grid links, we will reach $(4 - o(1))|N_g|$ more nodes within one more hop. If bad luck strikes, it could still happen that many of these links lead to a few nodes, already visited nodes, or nodes that are very close to each other. But that is very unlikely, as we have lots of random choices! Indeed, it can be shown that not only in expectation, but with high probability $(5 - o(1))|N_g|$ many nodes are reached this way.

Because all these shiny new nodes have (so far unused) random links, we can repeat this reasoning inductively, implying that the number of nodes grows by (at least) a constant factor for every two hops. Thus, after $O(\log n)$ hops, we will have reached $n/\log n$ nodes (which is still small compared to $n$). Finally, consider the expected number of links from these nodes that enter the $\log n$-neighborhood of some target node $w$ with respect to the grid. Since this neighborhood consists of $\Omega(\log^2 n)$ nodes, in expectation $\Omega(\log n)$ links come close enough to $w$. This is large enough to almost guarantee that this happens. Summing everything up, we still used merely $O(\log n)$ hops in total to get from $v$ to $w$.

This shows that for $\alpha = 0$ (and in fact for all $\alpha \leq 2$), the resulting network has a small diameter. Recall however that we also wanted the network to be navigable. For this, we consider a simple greedy routing strategy (Algorithm 44).

**Algorithm 44** Greedy Routing

1: while not at destination do
2: go to a neighbor which is closest to destination (considering grid distance only)
3: end while

**Lemma 12.5.** In the augmented grid, Algorithm 44 finds a routing path of length at most $2(m - 1) \in O(\sqrt{n})$.

**Proof.** Because of the grid links, there is always a neighbor which is closer to the destination. Since with each hop we reduce the distance to the target at least by one in one of the two grid dimensions, we will reach the destination within $2(m - 1)$ steps.
This is not really what Milgram’s experiment promises. We want to know how much the additional random links speed up the process. To this end, we first need to understand how likely it is that two nodes $u$ and $v$ are connected by a random link in terms of $n$ and their distance $d(u, v)$.

**Lemma 12.6.** Node $u$’s random link leads to a node $v$ with probability

- $\Theta(1/(d(u, v)^\alpha m^{2-\alpha}))$ if $\alpha < 2$.
- $\Theta(1/(d(u, v)^2 \log n))$ if $\alpha = 2$.
- $\Theta(1/d(u, v)^\alpha)$ if $\alpha > 2$.

Moreover, if $\alpha > 2$, the probability to see a link of length at least $d$ is in $\Theta(1/d^{\alpha-2})$.

**Proof.** For $\alpha \neq 2$, we have that

$$\sum_{w \in V \setminus \{u\}} \frac{1}{d(u, w)^\alpha} \in \sum_{r=1}^{m} \frac{\Theta(r)}{r^{\alpha}} = \Theta \left( \int_{r=1}^{m} \frac{1}{r^{\alpha-1}} \, dr \right) = \Theta \left( \frac{r^{2-\alpha}}{2-\alpha} \right)_1^m.$$

If $\alpha < 2$, this gives $\Theta(m^{2-\alpha})$, if $\alpha > 2$, it is in $\Theta(1)$. If $\alpha = 2$, we get

$$\sum_{w \in V \setminus \{u\}} \frac{1}{d(u, w)^2} \in \sum_{r=1}^{m} \frac{\Theta(r)}{r^2} = \Theta(1) \cdot \sum_{r=1}^{m} \frac{1}{r} = \Theta(\log m) = \Theta(\log n).$$

Multiplying with $d(u, v)^\alpha$ yields the first three bounds.

For the last statement, compute

$$\sum_{w \in V \atop d(u, w) \geq d} \Theta(1/d(u, v)^\alpha) = \Theta \left( \int_{r=d}^{m} \frac{r}{r^{\alpha}} \, dr \right) = \Theta \left( \frac{r^{2-\alpha}}{2-\alpha} \right)_d^m = \Theta(1/d^{\alpha-2}).$$

$\square$

**Remarks:**

- For $\alpha \neq 2$, this is bad news for the greedy routing algorithm, as it will take $n^{\Omega(1)} = nr^{\Omega(1)}$ expected steps to reach the destination. This is disappointing, we were hoping for something polylogarithmic.

- If $\alpha < 2$, in distance $m^{(2-\alpha)/3}$ to the target are $m^{3(2-\alpha)/3}$ many nodes. Thus it takes $\Theta(m^{(2-\alpha)/3})$ links in expectation to find a link that comes that close to the destination. Without finding such a link, we have to go at least this far using grid links only.

- If $\alpha > 2$, it takes $\Theta(m^{(\alpha-2)/(\alpha-1)})$ steps until we see a link of length at least $m^{1/(\alpha-1)}$ in expectation. Without such links, it takes at least $m/m^{1/(\alpha-1)} = m^{(\alpha-2)/(\alpha-1)}$ steps to travel a distance of $m$.

- Any algorithm that uses only the information on long-range contacts that it can collect at the so far visited nodes cannot be faster.

- However, the case $\alpha = 2$ looks more promising.
12.2. PROPAGATION STUDIES

Definition 12.7 (Phase). Consider routing from a node \( u \) to a node \( v \) and assume that we are at some intermediate node \( w \). We say that we are in phase \( j \) at node \( w \) if the lattice distance \( d(w, v) \) to the target node \( v \) is between \( 2^j \leq d(w, v) \leq 2^{j+1} \).

Remarks:

- Enumerating the phases in decreasing order is useful, as notation becomes less cumbersome.
- There are \( \lceil \log m \rceil \in O(\log n) \) phases.

Lemma 12.8. Assume that we are in phase \( j \) at node \( w \) when routing from \( u \) to \( v \). The probability for getting to phase \( j - 1 \) in one step is at least \( \Omega(1/\log n) \).

Proof. Let \( B_j \) be the set of nodes \( x \) with \( d(x, v) \leq 2^j \). We get from phase \( j \) to phase \( j - 1 \) if the long-range contact of node \( w \) points to some node in \( B_j \). Note that we always make progress while following the greedy routing path. Therefore, we have not seen node \( w \) before and the long-range contact of \( w \) points to a random node that is independent of anything seen on the path from \( u \) to \( w \).

For all nodes \( x \in B_j \), we have \( d(w, x) \leq d(w, v) + d(x, v) \leq 2^{j+1} + 2^j < 2^{j+2} \). Hence, for each node \( x \in B_j \), the probability that the long-range contact of \( w \) points to \( x \) is \( \Omega(1/2^{2j+4} \log n) \). Further, the number of nodes in \( B_j \) is at least \( (2^j)^2/2 = 2^{2j-1} \).

Hence, the probability that some node in \( B_j \) is the long range contact of \( w \) is at least

\[
\Omega \left( |B_j| \cdot \frac{1}{2^{2j+4} \log n} \right) = \Omega \left( \frac{2^{2j-1}}{2^{2j+4} \log n} \right) = \Omega \left( \frac{1}{\log n} \right).
\]

\( \square \)

Theorem 12.9. Consider the greedy routing path from a node \( u \) to a node \( v \) on an augmented grid with parameter \( \alpha = 2 \). The expected length of the path is \( O(\log^2 n) \).

Proof. We already observed that the total number of phases is \( O(\log n) \) (the distance to the target is halved when we go from phase \( j \) to phase \( j - 1 \)). At each point during the routing process, the probability of proceeding to the next phase is at least \( \Omega(1/\log n) \).

Let \( X_j \) be the number of steps in phase \( j \). Because the probability for ending the phase is \( \Omega(1/\log n) \) in each step, in expectation we need \( O(\log n) \) steps to proceed to the next phase, i.e., \( E[X_j] \in O(\log n) \). Let \( X = \sum_j X_j \) be the total number of steps of the routing process. By linearity of expectation, we have

\[
E[X] = \sum_j E[X_j] \in O(\log^2 n).
\]

\( \square \)

12.2 Propagation Studies

In networks, nodes may influence each other’s behavior and decisions. There are many applications where nodes influence their neighbors, e.g., they may impact their opinions, or they may bias what products they buy, or they may pass on a disease.

On a beach (modeled as a line segment), it is best to place an ice cream stand right in the middle of the segment, because you will be able to “control” the beach most easily. What about the second stand, where should it settle? The answer generally depends on the model, but assuming that people will buy ice cream from the stand that is closer, it should go right next to the first stand.
Rumors can spread astoundingly fast through social networks. Traditionally this happens by word of mouth, but with the emergence of the Internet and its possibilities new ways of rumor propagation are available. People write email, use instant messengers or publish their thoughts in a blog. Many factors influence the dissemination of rumors. It is especially important where in a network a rumor is initiated and how convincing it is. Furthermore the underlying network structure decides how fast the information can spread and how many people are reached. More generally, we can speak of diffusion of information in networks. The analysis of these diffusion processes can be useful for viral marketing, e.g. to target a few influential people to initiate marketing campaigns. A company may wish to distribute the rumor of a new product via the most influential individuals in popular social networks such as Facebook. A second company might want to introduce a competing product and has hence to select where to seed the information to be disseminated. Rumor spreading is quite similar to our ice cream stand problem.

More formally, we may study propagation problems in graphs. Given a graph, and two players. Let the first player choose a seed node $u_1$; afterwards let the second player choose a seed node $u_2$, with $u_2 \neq u_1$. The goal of the game is to maximize the number of nodes that are closer to one’s own seed node.

In many graphs it is an advantage to choose first. In a star graph for instance the first player can choose the center node of the star, controlling all but one node. In some other graphs, the second player can at least score even. But is there a graph where the second player has an advantage?

**Theorem 12.10.** In a two player rumor game where both players select one node to initiate their rumor in the graph, the first player does not always win.

**Proof.** See Figure 12.3 for an example where the second player will always win, regardless of the decision the first player. If the first player chooses the node $x_0$ in the center, the second player can select $x_1$. Choice $x_1$ will be outwitted by $x_2$, and $x_2$ itself can be answered by $z_1$. All other strategies are either symmetric, or even less promising for the first player. \qed
Chapter 13

Appendix: Overlay Design

This appendix provides a deeper look into the design of dynamic networks. It is not an official part of this year’s lecture, and just serves as a reference for the interested student. The original version of this text is by taken from the lecture by Prof. Christian Scheideler from Uni Paderborn.

13.1 Supervised Overlay Networks

Every application run on multiple machines needs a mechanism that allows the machines to exchange information. An easy way of solving this problem is that every machine knows the domain name or IP address of every other machine. While this may work well for a small number of machines, large-scale distributed applications such as file sharing or grid computing systems need a different, more scalable approach: instead of forming a clique (where everybody knows everybody else), each machine should only be required to know some small subset of other machines. This graph of knowledge can be seen as a logical network interconnecting the machines, which is also known as an overlay network. A prerequisite for an overlay network to be useful is that it has good topological properties. Among the most important are:

- **Degree**: Ideally, the degree should be kept small to avoid a high update cost if a node enters or leaves the system.
- **Diameter**: The diameter should be small to allow the fast exchange of information between any pair of nodes in the network.
- **Expansion**: The expansion of a graph $G = (V, E)$ is defined as

$$
\beta(G) = \min_{U \subseteq V: |U| \leq |V|/2} \frac{\left| \Gamma(U) \right|}{|U|}
$$

where $\Gamma(U)$ is the set of neighbors of $U$. To ensure a high fault tolerance, the expansion should be as large as possible.

The question is how to realize such an overlay network in a distributed environment where peers may continuously enter and leave the system. This will be the topic of our investigations for the coming weeks.

We start in this section with the study of supervised overlay networks. In a supervised overlay network, the topology is under the control of a special machine (or
node) called the supervisor. All nodes that want to join or leave the network have to declare this to the supervisor, and the supervisor will then take care of integrating them into or removing them from the network. All other operations, however, may be executed without involving the supervisor. In order for a supervised network to be highly scalable, two central requirements have to be fulfilled:

1. The supervisor needs to store at most a polylogarithmic amount of information about the network at any time (i.e., if there are \( n \) nodes in the network, storing contact information about \( O(\log^2 n) \) of these nodes would be fine, for example), and

2. it takes at most a constant number of communication rounds to include a new node into or exclude an old node from the network.

A communication round is over once all the packets that existed at the beginning of the communication round have been delivered. The packets generated by these packets will participate in the next communication round.

We show in the following how these requirements can be achieved, using a general approach called the recursive labeling approach. To simplify the presentation, we assume that all departures are graceful, i.e., every node leaving the system informs the supervisor about this and may provide some additional information simplifying the task of the supervisor to repair the network.

### 13.1.1 The Recursive Labeling Approach

In the recursive labeling approach, the supervisor assigns a label to every node that wants to join the system. The labels are represented as binary strings and are generated in the following order:

\[
0, 1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots
\]

Basically, when stripping off the least significant bit, then the supervisor is first creating all binary numbers of length 0, then length 1, then length 2, and so on. More formally, consider the mapping \( \ell : \mathbb{N}_0 \to \{0, 1\}^* \) with the property that for every \( x \in \mathbb{N}_0 \) with binary representation \( (x_d \ldots x_0)_2 \) (where \( d \) is minimum possible),

\[
\ell(x) = (x_{d-1} \ldots x_0 x_d).
\]

Then \( \ell \) generates the sequence of labels displayed above. In the following, it will also be helpful to view labels as real numbers in \([0, 1)\). Let the function \( r : \{0, 1\}^* \to [0, 1) \) be defined so that for every label \( \ell = (\ell_1 \ell_2 \ldots \ell_d) \in \{0, 1\}^* \),

\[
r(\ell) = \sum_{i=1}^{d} \frac{\ell_i}{2^i}.
\]

Then the sequence of labels above translates into

\[
0, 1/2, 1/4, 3/4, 1/8, 3/8, 5/8, 7/8, 1/16, 3/16, 5/16, 7/16, 9/16, \ldots
\]

Thus, the more labels are used, the more densely the \([0, 1)\) interval will be populated. Furthermore, we will use the function \( b : [0, 1) \to \{0, 1\}^* \) that translates a real number back into a label.

When using the recursive labeling approach, the supervisor aims to maintain the following condition at every step:
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Condition 13.1. The set of labels used by the nodes is \( \{\ell(0), \ell(1), \ldots, \ell(n-1)\} \), where \( n \) is the current number of nodes in the system.

This condition is preserved when using the following simple strategy:

- Whenever a new node \( v \) joins the system and the current number of nodes is \( n \), the supervisor assigns the label \( \ell(n) \) to \( v \) and increases \( n \) by 1.
- Whenever a node \( w \) with label \( \ell \) wants to leave the system, the supervisor asks the node with currently highest label \( \ell(n-1) \) to change its label to \( \ell \) and reduces \( n \) by 1.

How does this strategy help us with maintaining dynamic overlay networks? We will see how this works in the following subsections. To keep things simple, we start with a cycle.

13.1.2 Recursively Maintaining a Cycle

We start with some notation. In the following, the label assigned to some node \( v \) will be denoted as \( \ell_v \). Given \( n \) nodes with unique labels, we define the predecessor \( \text{pred}(v) \) of node \( v \) as the node \( w \) for which \( r(\ell_w) \) is closest from below to \( r(\ell_v) \), and we define the successor \( \text{succ}(v) \) of node \( v \) as the node \( w \) for which \( r(\ell_w) \) is closest from above to node \( r(\ell_v) \) (viewing \([0, 1)\) as a ring in both cases). Given two nodes \( v \) and \( w \), we define their distance as

\[
\delta(v, w) = \min\{(1 + r(\ell_v) - r(\ell_w)) \mod 1, (1 + r(\ell_w) - r(\ell_v)) \mod 1\}.
\]

In order to maintain a cycle among the nodes, we simply have to maintain the following condition:

Condition 13.2. Every node \( v \) in the system is connected to \( \text{pred}(v) \) and \( \text{succ}(v) \).

Now, suppose that the labels of the nodes are generated via the recursive strategy above. Then we have the following properties:

Lemma 13.3. Let \( n \) be the current number of nodes in the system, and let \( \bar{n} = 2^{\lceil \log n \rceil} \). Then for every node \( v \in V \):

- \( |\ell_v| \leq \lceil \log n \rceil \) and
- \( \delta(v, \text{pred}(v)) \in [1/(2\bar{n}), 1/\bar{n}] \) and \( \delta(v, \text{succ}(v)) \in [1/(2\bar{n}), 1/\bar{n}] \).

So the nodes are approximately evenly distributed in \([0, 1)\) and the number of bits for storing a label is almost as low as it can be without violating the uniqueness requirement. But how does the supervisor maintain the cycle? This is implied by the following demand, where \( n \) is again the current number of nodes in the system.

Condition 13.4. At any time, the supervisor stores the contact information of \( \text{pred}(v) \), \( v \), \( \text{succ}(v) \), and \( \text{succ}(\text{succ}(v)) \) where \( v \) is the node with label \( \ell(n-1) \).

In order to satisfy Conditions 13.2 and 13.4, the supervisor performs the following actions, where \( v \) is the node with label \( \ell(n-1) \) in the system.

If a new node \( w \) joins, then the supervisor
• informs \( w \) that \( \ell(n) \) is its label, \( \text{succ}(v) \) is its predecessor, and \( \text{succ}(\text{succ}(v)) \) is its successor,
• informs \( \text{succ}(v) \) that \( w \) is its new successor,
• informs \( \text{succ}(\text{succ}(v)) \) that \( w \) is its new predecessor,
• asks \( \text{succ}(\text{succ}(v)) \) to send its successor information to the supervisor, and
• sets \( n = n + 1 \).

If an old node \( w \) leaves and reports \( \ell_w, \text{pred}(w), \text{succ}(w) \) to the supervisor (recall that we are assuming graceful departures), then the supervisor
• informs \( v \) (the node with label \( \ell(n - 1) \)) that \( \ell_w \) is its new label, \( \text{pred}(w) \) is its new predecessor, and \( \text{succ}(w) \) is its new successor,
• informs \( \text{pred}(w) \) that its new successor is \( v \),
• informs \( \text{succ}(w) \) that its new predecessor is \( v \),
• informs \( \text{pred}(v) \) that \( \text{succ}(v) \) is its new successor,
• informs \( \text{succ}(v) \) that \( \text{pred}(v) \) is its new predecessor,
• asks \( \text{pred}(v) \) to send its predecessor information to the supervisor and to ask \( \text{pred}(\text{pred}(v)) \) to send its predecessor information to the supervisor, and
• sets \( n = n - 1 \).

A detailed implementation of the leave and join operations can be found in Figures 13.1 and 13.2. In this implementation, we assume for simplicity that references to relay points can be freely exchanged, i.e., identities are not needed. It will be an assignment to implement the join and leave operations with the identity concept. The following lemma is not difficult to check and will also be an assignment.

Lemma 13.5. The join and leave operations preserve Conditions 13.2 and 13.4.

Hence, we arrive at the following theorem, which implies that our central requirements on a supervisor are kept.

Theorem 13.6. At any time, the supervisor only needs to store the current value of \( n \) and a constant amount of contact information, and the join and leave operations only need a constant amount of messages and three communication rounds to complete.

13.1.3 Concurrency

The above scheme only allows the supervisor to execute join and leave operations in a strictly sequential manner because it only has sufficient information to integrate or remove one peer at a time. In order to be able to handle \( d \) join or leave requests in parallel, we extend Condition 13.2 with the following rule:

Condition 13.7. In addition to Condition 13.2, every node \( v \) in the system is connected to its \( d \)th predecessor \( \text{pred}_d(v) \) and its \( d \)th successor \( \text{succ}_d(v) \).

Furthermore, given that \( v \) is the node with label \( \ell(n - 1) \), Condition 13.4 needs to be extended to:
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Supervisor \{(\text{Supervisor}) \{ \\
n := 0 \quad \# \text{counter} \\
v := \text{NULL} \quad \# \text{node with label } \ell(n - 1) \\
pv := \text{NULL} \quad \# \text{pred}(v) \\
sv := \text{NULL} \quad \# \text{succ}(v) \\
ssv := \text{NULL} \quad \# \text{succ(succ}(v)) \\
\}} \\
\text{Join}(w; \text{Relay}) \{ \\
\text{if } (n = 0) \{ \\
w \leftarrow \text{setup}(0, w, w) \\
pv \leftarrow w \\
v \leftarrow w \\
sv \leftarrow w \\
ssv \leftarrow w \\
\} \text{ else } \{ \\
w \leftarrow \text{setup}(\ell(n), sv, ssv) \\
sv \leftarrow \text{setSucc}(w) \\
ssv \leftarrow \text{setPred}(w) \\
pv \leftarrow sv \\
v \leftarrow w \\
sv \leftarrow ssu \\
ssv \leftarrow \text{setSucc}(ssv) \\
n := n + 1 \\
\} \\
\text{Leave}(\ell; \text{Int}, pw; \text{Relay}, sw; \text{Relay}) \{ \\
\text{if } (n > 0) \{ \\
\text{if } (n = 1) \{ \\
pv := \text{NULL}, v := \text{NULL} \\
sv := \text{NULL}, ssu := \text{NULL} \\
\} \text{ else } \{ \\
\# \text{remove } v \text{ from the system} \\
pv \leftarrow \text{setSucc}(sv) \\
sv \leftarrow \text{setPred}(pv) \\
\text{if } (pv = v) \text{ pv } := \text{pv} \\
\text{if } (sv = v) \text{ sv } := \text{sv} \\
\# \text{move } v \text{ into position of } w \\
\text{if } (v \neq w) \{ \\
v \leftarrow \text{setup}(\ell, pv, sw) \\
pv \leftarrow \text{setSucc}(v) \\
sv \leftarrow \text{setPred}(v) \\
\} \\
\# \text{update pointers} \\
\text{if } (pv = w) \text{ pv } := v \\
\text{if } (sv = w) \text{ sv } := v \\
ssv := sv \\
sv := pv \\
v := pv \leftarrow \text{getPred}() \\
pv := pv \leftarrow \text{getPredPred}() \\
n := n - 1 \\
\} \\
\}

Figure 13.1: Operations needed by the supervisor to maintain a cycle.

**Condition 13.8.** At any time, the supervisor stores the contact information of \(v\), the 2d successors of \(v\), and the 3d predecessors of \(v\).

These conditions are preserved in the following way.

**Concurrent Join Operation.** In the following, let \(v\) be the node with label \(\ell(n - 1)\). Let \(\text{succ}_i(v)\) denote the \(i\)th successor of \(v\) on the cycle and \(\text{pred}_i(v)\) denote the \(i\)th predecessor of \(v\) on the cycle.

Let the \(d\) new peers be \(w_1, w_2, \ldots, w_d\). Then the supervisor integrates \(w_i\) between \(\text{succ}_i(v)\) and \(\text{succ}_{i+1}(v)\) for every \(i \in \{1, \ldots, d\}\). As is easy to check, this will violate Condition 13.7 for the 2d closest successors of \(v\) and the \(d - 2\) closest predecessors of \(v\). But since the supervisor knows all of these nodes, it can directly inform them about the change. In order to repair Condition 13.8, the supervisor will request information about the \(d\)th successor from the \(d\) furthest successors of \(v\) and will set \(v\) to \(w_d\).

**Concurrent Leave Operation.** Let the \(d\) peers that want to leave the system be \(w_1, w_2, \ldots, w_d\). For simplicity, we assume that they are outside of the peers known to the supervisor and that they are not in the neighborhood of each other, but our strategy


```
Peer() {
  label := 0  # label of peer v
  succ := NULL  # succ(v)
  pred := NULL  # pred(v)
  sr := new Relay()  # relay point of v
}

Join(s: Relay) {  # relay of supervisor
  if (s ≠ NULL) {
    s ← Join(sr)
    super := s  # current supervisor
  }
}

Leave() {
  if (super ≠ NULL)
    super ← Leave(label, pred, succ)
    super := NULL,
}

setup(ℓ : Int, p : Relay, s : Relay) {
  label := ℓ
  pred := p
  succ := s
}

setSucc(w: Relay) {
  succ := w
}

setPred(w: Relay) {
  pred := w
}

getSucc(): Relay {
  return succ
}

getPred(): Relay {
  return pred
}

getPredPred(): Relay {
  return pred ← getPred()
}
```

Figure 13.2: Operations needed by a peer to maintain a cycle.

below can also be extended to these cases. The strategy of the supervisor is to replace

\( w_i \) by \( \text{pred}_{2i-1}(v) \) for every \( i \). As is easy to check, this will violate Condition 13.7

for the \( d \) closest successors of \( v \) and the \( 3d \) closest predecessors of \( v \). But since the

supervisor knows all of these nodes, it can directly inform them about the change. In

order to repair Condition 13.7, the supervisor will request information about the \( d \)

predecessor from the \( d \) furthest predecessors of \( v \) and their \( d \)th predecessors and will

set \( v \) to \( \text{pred}_{2d}(v) \).

The operations have the following performance.

**Theorem 13.9.** The supervisor needs at most \( O(d) \) work and \( O(1) \) time (given that the

work can be done in parallel) to process \( d \) join or leave requests.

### 13.1.4 Multiple Supervisors

If a supervised network becomes so large that a single supervisor cannot manage all

of the join and leave requests, one can easily extend the supervised cycle to multiple

supervisors. Suppose that we have \( k \) supervisors \( S_0, S_1, \ldots, S_{k-1} \). Then the \([0, 1)\)

-ring is split into the \( k \) regions \( R_i = \left[(i-1)/k, i/k\right), 1 \leq i \leq k \), and supervisor \( S_i \)

is responsible for region \( R_i \). Every supervisor manages its region as described for a

single supervisor above, i.e., it treats it like a \([0, 1)\)-interval, except for the borders, and

the borders are maintained by communicating with the neighboring supervisors on the

ring. The supervisors themselves form a completely interconnected network.
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Each time a new node \( v \) wants to join the system via some supervisor \( S_i \), \( S_i \) forwards it to a random supervisor to integrate \( v \) into the system. Each time a node \( v \) under some supervisor \( S_i \) wants to leave the system, \( S_i \) replaces that node with the last node it inserted into \( R_i \). Using standard Chernoff bounds, we get:

**Theorem 13.10.** Let \( n \) be the total number of nodes in the system. If the join-leave behavior of the nodes is independent of their positions, then it holds for every \( i \in \{1, \ldots, k\} \) that the number nodes currently placed in \( R_i \) is in the range \( n/k \pm O(\sqrt{(n/k) \log k + \log k}) \), with high probability.

Hence, if \( n \) is sufficiently large compared to \( k \), then the multi-supervised cycle has basically the same properties as the single-supervised cycle above. If the join-leave behavior of the nodes is adversarial, then the rules of assigning every new node to the least loaded region \( R_i \) and replacing every leaving node with the node inserted last into the most loaded region \( R_i \) will keep a balanced distribution of the nodes among the regions.

13.1.5 Recursively Maintaining a Tree

The cycle has a low degree but its diameter and expansion are very bad. The simplest way of achieving a low diameter is to use a tree. Thus, next we discuss how to recursively maintain a tree. As for the cycle, our basic approach will be to preserve something similar to Condition 13.1, with the only difference that we want to keep the labels from \( \ell(1) \) to \( \ell(n) \) (instead of \( \ell(0) \) to \( \ell(n-1) \)). We will also preserve Condition 13.2, though the edges implied by this condition will not be part of the tree. But they will tremendously simplify the task of maintaining a tree, as we will see.

Recall that a binary tree can be stored in an array by connecting position \( x \) to positions \( 2x \) and \( 2x + 1 \) for any \( x \geq 1 \). In our context with node labels, this would mean that each node with label \( (\ell_1, \ldots, \ell_d) \) has to be connected to the nodes with labels \( (\ell_1, \ldots, \ell_{d-1}, x\ell_d) \) where \( x \in \{0, 1\} \) (see the way labels can be interpreted as binary numbers in the recursive labeling approach). Thus, the following connectivity information has to be preserved.

**Condition 13.11.** Every node \( v \) in the system with label \( \ell_v = (\ell_1, \ldots, \ell_d) \) is connected to

1. \( \text{pred}(v) \) and \( \text{succ}(v) \) (to form a cycle) and
2. the nodes with labels \( (\ell_1, \ldots, \ell_{d-1}01), (\ell_1, \ldots, \ell_{d-1}10), \) and \( (\ell_1, \ldots, \ell_{d-1}11) \), if they exist (to form a tree).

Suppose that this condition is kept at any time. Then the following lemma follows.

**Lemma 13.12.** At any time, the \( n \) nodes form a binary tree of depth \( \lceil \log n \rceil - 1 \).

**Proof.** Consider a binary tree with \( n \) nodes, and label the edge to the left child of any node “0” and to the right child of any node “1”. Let the label \( t_v \) of every node \( v \) in this tree be the sequence of edge labels when moving along the unique path from the root to \( v \). Then every node \( v \) with label \( (\ell_1, \ldots, \ell_d) \) is connected to the node with label \( (\ell_1, \ldots, \ell_{d-1}) \) (its parent), if it exists, and is also connected to the nodes with labels \( (\ell_1, \ldots, \ell_d0) \) and \( (\ell_1, \ldots, \ell_d1) \) (its children), if they exist. Defining \( t_v \) as \( \ell_v \) (the label of \( v \) in our network) without the least significant bit, we see that Condition 13.11(2) fulfills the connectivity requirements of a tree. Since it follows from Lemma 13.3 that
Every node has a label of size at most \( \lceil \log n \rceil \), the depth of the tree can be at most \( \lceil \log n \rceil - 1 \).

Next we specify the connectivity information the supervisor needs in order to maintain the tree.

**Condition 13.13.** At any time, the supervisor stores the contact information of \( \text{pred}(v) \), \( v \), \( \text{succ}(v) \), and \( \text{succ(succ}(v)) \) where \( v \) is the node with label \( \ell(n) \).

Hence, the supervisor does not need any further connectivity information beyond what it needs for the cycle. In order to satisfy Conditions 13.11 and 13.13, the supervisor performs the following actions. If a new node \( w \) joins, then the supervisor

- informs \( w \) that \( \ell(n+1) \) is its label, \( \text{succ}(v) \) is its predecessor, and \( \text{succ(succ}(v)) \) is its successor, and \( \text{succ}(v) \) resp. \( \text{succ(succ}(v)) \) is its parent (depending on \( \ell(n+1) \)),
- informs \( \text{succ}(v) \) that \( w \) is its new successor,
- informs \( \text{succ(succ}(v)) \) that \( w \) is its new predecessor,
- asks \( \text{succ(succ}(v)) \) to send its successor information to the supervisor, and
- sets \( n = n + 1 \).

Hence, from the point of view of the supervisor, the inclusion of a new node is almost identical to the cycle.

If an old node \( w \) leaves and reports \( \ell(w) \), \( \text{pred}(w) \), \( \text{succ}(w) \), \( \text{parent}(w) \), \( \text{lchild}(w) \), and \( \text{rchild}(w) \) to the supervisor, then the supervisor again executes almost the same steps as for the cycle.

When using the code for the supervisor given in Figure 13.3 and the code for the peers given in Figure 13.4, it is not difficult to prove the following lemma. Notice that for simplicity, we assume again that relay points can be freely exchanged.


Hence, we arrive at the following theorem.

**Theorem 13.15.** At any time, the supervisor only needs to store the current value of \( n \) and a constant amount of contact information, and the join and leave operations only need a constant amount of messages and three communication rounds to complete.

**Broadcasting**

The dynamic tree can be used for efficient broadcasting. Suppose that some node \( v \) wants to broadcast information to all other nodes in the system. One way of solving this is that it forwards the broadcast message directly to the supervisor (so that the supervisor can authorize the broadcast, for example) and the supervisor initiates sending the broadcast message down the tree. A prerequisite for this is that the supervisor remembers the node with label 1, called \( \text{root} \) by it. If this is the case, then the code in Figure 13.5 will be executed correctly.

Inspecting the code, we arrive at the following result, which is optimal for broadcasting in constant degree networks. Here, the **dilation** means the longest path taken by a message in the broadcast operation.
The broadcast operation has a dilation of $O(\log n)$ and requires a work of $O(n)$.

### Maintaining a fault-tolerant tree

Recall that in order to store a tree in an array, we connect position $x$ to positions $2x$ and $2x + 1$ for any $x \geq 1$. Such a tree can easily be made fault-tolerant by demanding that each position $x$ be connected to all positions in the set $\{2x, \ldots, 2(x + r) - 1\}$ for some parameter $r \in \mathbb{N}$ that we call its redundancy. If $r = 1$, we just arrive at the binary tree, but when choosing $r > 1$, each node has $r$ parents instead of just 1. Hence, as long as not all $r$ parents of an alive node are defunct, all alive nodes can still reach one of the $r$ topmost nodes in the array. Transforming to our use of node labels, we arrive at the following condition for the nodes.

**Condition 13.17.** For some fixed $r \in \mathbb{N}$, every node $v$ in the system with label $\ell_v = (\ell_1, \ldots, \ell_d)$ is connected to

1. its closest $r$ predecessors and successors in $[0, 1)$ (to form a redundant cycle) and

---

Figure 13.3: Operations needed by the supervisor to maintain a tree.

```plaintext
Supervisor {
  Supervisor() {
    n := 0 \quad \# counter
    v := NULL \quad \# node with label $\ell(n)$
    pv := NULL \quad \# pred(v)
    sv := NULL \quad \# succ(v)
    ssv := NULL \quad \# succ(succ(v))
  }

  Join(w: Relay) {
    n := n + 1
    if (n = 1) {
      w := setup(0, w, NULL, NULL, NULL)
      pv := w
      sv := w
      ssv := w
    } else {
      if ($\ell(n)$&2 = 0) {
        w := setup($\ell(n)$, sv, ssv, NULL, NULL)
        ssv := setRightChild(v)
      } else {
        w := setup($\ell(n)$, sv, ssv, sv, NULL, NULL)
        sv := setLeftChild(w)
      }
      sv := setSucc(w)
      ssv := setPred(w)
      pv := sv
      sv := sv
      ssv := sv
      ssv := getSucc()
    }
  }

  Leave(f: Int, pv: Relay, sv: Relay, f_w, l_w, r_w: Relay) {
    if (n > 0) {
      if (n = 1) {
        pv := NULL, v := NULL
        sv := NULL, ssv := NULL
      } else {
        # remove v from tree
        if ($\ell(n - 1)$&2 = 0) sv := setRightChild(NULL)
        else pv := setLeftChild(NULL)
        pv := setSucc(sv)
        sv := setPred(pv)
        if (pu = v) pv := pv
        if (su = v) sv := sv
        if (l_w = v) l_w := NULL
        if (r_w = v) r_w := NULL
        # move v into position of w
        if ($v \neq w$) {
          v := setup($\ell$, pv, sv, ssv, NULL, NULL)
          sv := setPred(v)
          sv := sw
          pv := setPred(pv)
          pv := pv
        } else {
          if ($\ell$&2 = 0) f_w := setRightChild(v)
          else f_w := setLeftChild(v)
          if (l_w = NULL) l_w := setPred(v)
          if (r_w = NULL) r_w := setPred(v)
        }
        # update pointers
        if (pu = v) pv := pv
        if (sv = w) sv := v
        ssv := sv
        sv := pv
        pv := pv
        pv := getPred()
        pv := pv
        pv := getPred()
      }
      n := n - 1
    }
  }
}
```
CHAPTER 13. APPENDIX: OVERLAY DESIGN

Peer { Peer() { 
  label := 0 # label of peer v 
  succ := NULL # succ(v) 
  pred := NULL # pred(v) 
  parent := NULL 
  lchild := NULL 
  rchild := NULL 
  sr := new Relay() # relay point of v 
}
}

Join(s: Relay) { 
  if (s ≠ NULL) { 
    s → Join(sr) 
    super := s # current supervisor 
  } 
}

Leave() { 
  if (super ≠ NULL) 
    super ← Leave(label, pred, succ, parent, lchild, rchild) 
    super := NULL 
}

setup(ℓ: Int, p: Relay, s: Relay, f: Relay, lc: Relay, rc: Relay) { 
  label := ℓ 
  pred := p 
  succ := s 
  parent := f 
  lchild := lc 
  rchild := rc 
}

setSucc(w: Relay) { succ := w }

setPred(w: Relay) { pred := w }

.setParent(w: Relay) { parent := w }

setLeftChild(w: Relay) { lchild := w }

setRightChild(w: Relay) { rchild := w }

getSucc(): Relay { return succ }

getPred(): Relay { return pred }

getPredPred(): Relay { return pred ← getPred() }

Figure 13.4: Operations needed by a peer to maintain a tree.

2. all nodes w with labels (ℓ_1′, . . . , ℓ_d′) so that for x′ = (ℓ_1′, ℓ_2′, . . . , ℓ_{d-1}′) and x = (ℓ_1, ℓ_2, . . . , ℓ_{d-1}) it holds that x′ ∈ {x − r + 1, . . . , x} (w is one of the parents of v) or x′ ∈ {2x, . . . , 2(x + r) − 1} (w is one of the children of v).

The supervisor has to maintain the following connections to efficiently update such a tree.

Condition 13.18. At any time, the supervisor stores the contact information of v and its 2r closest predecessors and its 2 closest successors, where v is the node with label ℓ(n).

2r predecessors are needed to keep track of the r parents of a tree node, and r successors are needed (with the predecessors) to maintain a redundant ring. As mentioned above, this structure can tolerate many defunct nodes without running into problems when broadcasting information between the alive nodes. More details are left to the reader.

13.1.6 Recursively Maintaining a de Bruijn Graph

Next, we show how to maintain a supervised de Bruijn network. Recall the definition of a de Bruijn graph. In this definition, every node with label (x_1, . . . , x_d) ∈ \{0, 1\}^d
is connected to the nodes \((0, x_1, \ldots, x_d)\) and \((1, x_1, \ldots, x_d)\). When interpreting every node with label \((x_1, \ldots, x_d)\) as a point \(x = \sum_{i \geq 1} x_i/2^i \in [0, 1)\) and letting \(d \to \infty\), we arrive at the following continuous form of the de Bruijn graph:

- \(U = [0, 1)\)
- \(F = \{\{x, y\} \in U^2 | f_0(x) = y \text{ or } f_1(x) = y\}\)

Now, recall the way in which the nodes in consistent hashing partitioned the \([0, 1)\)-interval among them. We can use a similar strategy here. Suppose that each node \(v\) with position \(x_v \in [0, 1)\) is given the interval \(I_v = [x_v, x_{\text{succ}(v)})\) (considering \([0, 1)\) as a ring here). Then we have the property that \(\bigcup_v I_v = [0, 1)\) and, due to Lemma 13.3, \(|I_v| \in [1/(2n), 1/n]\) for every node \(v\). Suppose now that nodes maintain the following condition:

**Condition 13.19.** Every node \(v\) in the system is connected to

- \(\text{pred}(v)\) and \(\text{succ}(v)\) (in order to form a circle) and
- all nodes \(w\) with \(I_w \cap (f_0(I_v) \cup f_1(I_v) \cup f_0^{-1}(v) \cup f_1^{-1}(v)) \neq \emptyset\) (in order to be able to emulate the continuous de Bruijn graph).

Then the nodes in our system can emulate any message transmission along an edge \(\{x, y\} \in F\) since for any such edge there must be two nodes \(v\) and \(w\) in our system with \(x \in I_v\) and \(y \in I_w\), and these nodes must be connected due to the condition above. When combining Condition 13.19 with our recursive labeling approach, the following result holds:

**Theorem 13.20.** At any time, the supervised de Bruijn network has a degree of \(O(1)\), a diameter of \(O(\log n)\) and an expansion of \(\Omega(1/\log n)\), where \(n\) is the number of peers in the system.
Hence, the emulation the continuous de Bruijn graph yields a well-connected, low-degree graph for the peers that is, in fact, close to an ideal de Bruijn graph. Consider, for example, the problem of routing a message from node \( v \) to node \( w \), and suppose that \( v \) knows \( x_w \). Let \( x_v = (x_1, x_2, x_3, \ldots) \) and \( x_w = (y_1, y_2, y_3, \ldots) \) (i.e., \( x_v = \sum_{i \geq 1} x_i / 2^i \)). Then \( v \) may select a random intermediate point \( z = (z_1, z_2, z_3, \ldots) \in [0, 1) \) (like in Valiant’s trick). \( v \) first routes its message along the nodes owning the points \((x_1, x_2, x_3, \ldots), (z_1, z_2, z_3, \ldots), (z_2, z_1, x_1, x_2, \ldots), \) and so on, until it reaches a node \( u \) in which the two points \((z_k, \ldots z_1, y_1, y_2, \ldots), (z_{k-1}, \ldots, z_1, y_1, y_2, \ldots)\) are either both in \( I_u \) or one is in \( I_u \) while the other is in one of its neighboring intervals (which is true w.h.p. for \( k = O(\log n) \)). Afterwards, the message is sent along the node owning the point \((y_1, y_2, y_3, \ldots)\). Altogether, this just takes \( O(\log n) \) communication rounds.

When using the same supervisor strategy as for the supervised cycle (the supervisor introduces a new node to its neighbors in \([0, 1)\)), then Condition 13.19 implies that the predecessor of the new node \( v \) has all the connectivity information \( v \) needs to get fully integrated into the network. On the other hand, if an old node \( u \) wants to leave the system, and \( u \) is replaced by the node with largest label \( v \), then \( \text{pred}(v) \) just takes over all of the connections of \( v \) and \( v \) takes over all connections of \( u \) in order to satisfy Condition 13.19 after the removal of \( u \). This gives the following theorem.

**Theorem 13.21.** Using our framework, the supervisor can maintain a dynamic de Bruijn network with work and time \( O(1) \) for each join and leave request.

### 13.1.7 Applications

Finally, we discuss some applications of the supervised overlay networks that arise in the area of distributed computing.

**Grid Computing**

Recently, many systems such as SETI@home, Folding@home, and Distributed.net have been proposed for distributed computing. A main drawback of such systems is that the topology of the system is a star graph with the central server maintaining a direct connection to each client. Such a topology imposes heavy demands on the central server. Instead, we can use our framework for supervised overlay networks to maintain an overlay network for distributed computing. Peer-to-peer connections allow subtasks to be spawned without the involvement of the supervisor so that the demands on the server can be significantly reduced. This is particularly interesting for distributed branch-and-bound computations.

**WebTv**

Our approach can also be used in Internet applications such as WebTv. In such an application, there are typically various channels that users can browse or watch while being connected to the Internet. The number of channels ranges in the scale of hundreds while the number of users can range in the scale of millions. Such a system should allow users to quickly zap through channels. Hence, such a system should allow for rapid integration and be scalable to a large number of users. Our supervised overlay networks can easily achieve such a smooth operation. Suppose that every channel has a supervisor, each supervisor maintains its own broadcast network, and the supervisors
form a clique. Then it follows from our supervised approach, which can handle join and leave operations in constant time, that users browsing through channels can be moved between the networks in a very fast way, comparable to server-based networks, so that users only experience an insignificant delay.

Massive Multi-player Online Gaming

Distributed architectures for massive multi-player online gaming (MMOG) have only recently been studied formally. The basic requirements of such a system includes authentication, scalability, and rapid integration. Traditionally, such systems have been managed by a central server that takes care of the overall system with limited communication between the users. Certainly, such a system will not be scalable and also might experience heavy congestion at the central server. Hence, distributed architectures are required at a certain scale. A supervised overlay network approach can help here. For example, in a large virtual world, every supervisor may be responsible for a certain part of the world, and the supervisors may be interconnected like a cellular network to allow a fast handover process between them. Each supervisor then takes care of the peers currently exploring its part of the world. Since in our supervised approach peers can quickly be integrated and removed from a network, the handover process can be realized in a very fast way so that even fast moving peers can be handled. Additional supervisors may also be used for load balancing purposes in a sense that whenever a supervisor is heavily loaded, other supervisors may help out by taking over some of its peers and/or parts of the virtual world. In this way, it should be possible to create new generations of games in very complex worlds.

13.2 Decentralized Overlay Networks

In the next two sections we present overlay networks that are completely decentralized, i.e., they do not depend on a supervisor. We assume that, in principle, every peer has the right to initiate the integration of new peers into the system and that every peer knows at least one peer currently in the system so that publicly available entry points such as a supervisor are not necessary any more. In this section, we will focus on overlay networks that are based on the continuous-discrete approach, and in the next section overlay networks are presented that are based on so-called skip graphs. First, we will first assume that all peers are reliable and honest, and later we will show how to remove this assumption.

13.2.1 Virtual Space Management

Many decentralized peer-to-peer systems are based on the concept of a virtual space. That is, we are given a space $U$ and every peer $v$ is associated with a region $R(v) \subseteq U$ so that $\bigcup_{v \in V} R(v) = U$ for the current set of peers $V$. This property has to be maintained while peers join and leave the system. A very general concept for doing this is the hierarchical decomposition approach (recall Section 3 or 5).

Hierarchical Decomposition

We assume that there is a generic way of recursively cutting $U$ in half. In order to simplify the presentation, we assume that $U = [0, 1)^d$ for some fixed $d \geq 1$. The
**decomposition tree** \( T(U) \) of \( U \) is an infinite binary tree in which the root represents \( U \) and for every node \( v \) representing a subcube \( U' \) in \( U \), the children of \( v \) represent two subcubes \( U'' \) and \( U''' \), where \( U'' \) and \( U''' \) are the result of cutting \( U' \) in the middle at the smallest dimension in which \( U' \) has a maximum side length. The subcubes \( U'' \) and \( U''' \) are closed, i.e., their intersection gives the cut. Let every edge to a left child in \( T(U) \) be labeled with 0 and every edge to a right child in \( T(U) \) be labeled with 1. Then the label of a node \( v \), \( \ell(v) \), is the sequence of all edge labels encountered when moving along the unique path from the root of \( T(U) \) downwards to \( v \). For \( d = 2 \), the result of this decomposition is shown in Figure 13.6.

Figure 13.6: The decomposition tree for \( d = 2 \).

The goal is to map the peers to nodes in \( T(U) \) so that the following conditions are met:

**Condition 13.22.**

1. The interiors of the subcubes associated with the (tree nodes assigned to the) peers are disjoint,
2. the union of the subcubes of the peers gives the entire set \( U \).

In order to preserve this condition, the following is done when peers join or leave the system.

**Joining the System**

When a new peer \( p \) joins the system, it follows down the decomposition tree (which is simulated by a proper routing scheme in the given overlay network) according to its (random or pseudo-random) label \( \ell(p) \) until it arrives at some node \( v \) that is currently occupied by peer \( q \). Then \( q \) is moved to node \( v0 \) and \( p \) is moved to node \( v1 \) of the decomposition tree (i.e., \( q \) splits its region into two pieces and gives one of them to \( p \)) so that Condition 13.22 still holds.

**Leaving the System**

When a peer \( p \) at node \( v \) in the decomposition tree leaves, the peer at the lowest position in the decomposition tree that is reachable from the sibling of \( v \), say \( q \), is taken to replace the position of \( p \) (and thereby takes over its region). \( q \) must have a peer \( q' \) at its
prior sibling position in the decomposition tree which is moved to its former parent to take over q’s old region.

Given the two rules, one can show the following result for $U = [0, 1]^d$.

**Lemma 13.23.** Suppose that there are $n$ peers in the system. If the join-leave activity of the peers is independent of their labels, then the level of every peer in the decomposition tree is within $\log n \pm (\log \log n + O(1))$, w.h.p.

The lemma implies that the sizes of the regions assigned to the peers only differ by a factor of $O(\log n)$. Often, using the decomposition tree approach is overly complicated, especially when $U = [0, 1)$. In this case, a much simpler strategy is to use the consistent hashing approach in order to partition $[0, 1)$ among the peers:

- Every peer $p$ is assigned to some random point $x_p \in [0, 1)$.
- Every peer $p$ is responsible for the region $R_p = [x_p, \text{succ}(x_p))$ where $\text{succ}(x_p)$ is the closest point succeeding $x_p$ in $[0, 1)$ that is occupied by a peer.

Using this rule, it is obvious that the regions are pairwise disjoint and that $\bigcup R_v = [0, 1)$, which is necessary for applying the continuous-discrete approach. For this strategy, it holds:

**Lemma 13.24.** Suppose that there are $n$ peers in the system. If the join-leave activity of the peers is independent of their positions, then every peer is responsible for a region of size at least $\Omega(1/n^3)$ and most $O(\log n/n)$, w.h.p.

**Proof.** We first prove the upper bound. Consider any interval $I$ of size $(c \ln n)/n$ for some sufficiently large constant $c > 0$. The probability that none of the peers has its point in $I$ is equal to

$$
\left(1 - \frac{c \ln n}{n}\right)^n \leq e^{-\frac{(c \ln n)}{n}} = e^{-c \ln n} = n^{-c}.
$$

Hence, when partitioning $[0, 1)$ into $n/(c \log n)$ such intervals, every one of these has at least one point in them, w.h.p. Thus, a peer can be responsible for a region of size at most $O(\log n/n)$, w.h.p.

Next we prove the lower bound. The probability that any two peer positions have a distance of less than $1/n^3$ is at most

$$
\binom{n}{2} \frac{1}{n^3} \leq \frac{1}{2n}
$$

Hence, the probability is very low that such a case occurs, completing the proof.

The upper bound is acceptable though much better results can be achieved when performing local load balancing. In fact, evenly balancing the region size among the $\Theta(\log n)$ closest peers in $[0, 1)$ would ensure that the region size of every peer is $\Theta(1/n)$, w.h.p., which is implied by the following lemma.

**Lemma 13.25.** Given $n$ peers, every interval of size $\Theta(\log n/n)$ has $\Theta(\log n)$ peers in it, w.h.p.
Proof. Consider some fixed interval $I$ of size $(c \ln n)/n$ for some sufficiently large constant $c > 0$. For every peer $v$ let the binary random variable $X_v$ be 1 if and only if $x_v \in I$. Let $X = \sum_{v \in V} X_v$. It holds that

$$E[X_v] = \Pr[X_v = 1] = \frac{c \ln n}{n}$$

and from the linearity of expectation it follows that

$$E[X] = \sum_{v \in V} E[X_v] = n \cdot \frac{c \ln n}{n} = c \ln n.$$ 

Hence, when using the well-known Chernoff bounds, we obtain that

$$\Pr[X \geq (1 + \epsilon)E[X]] \leq e^{-c^2E[X]/3} = e^{-c^2c \ln n/3} = n^{-c^2c/3}$$

and

$$\Pr[X \leq (1 - \epsilon)E[X]] \leq e^{-c^2E[X]/2} = n^{-c^2c/3}$$

for all $0 \leq \epsilon \leq 1$. Thus, the probability is polynomially small in $n$ that the bound in the lemma is violated. \qed

13.2.2 The Continuous-Discrete Approach

The basic idea underlying the continuous-discrete approach is to define a continuous model of graphs and to apply this continuous model to the discrete setting of a finite set of peers. A well-known peer-to-peer system that uses an approach closely related to the continuous-discrete approach is Chord.

Consider any space $U$, and suppose that we have a (possibly infinite) collection $F$ of functions $f_i : U \rightarrow U$. Let

$$E_F = \{\{x, y\} \in U^2 \mid \exists i : y = f_i(x)\}$$

Then $(U, E_F)$ can be seen as an undirected graph on an infinite number of nodes. For any set $S \subseteq U$ let $\Gamma(S) = \{y \in U \setminus S \mid \exists x \in S : \{x, y\} \in E_F\}$ be the neighbor set of $S$ (i.e., all points $y$ with $y = f_i(x)$ or $x = f_i(y)$ for some $i$ since we consider undirected edges). If $\Gamma(S) \neq \emptyset$ for every $S \subseteq U$, then $F$ is said to be mixing. If $F$ does not mix, then there are disconnected areas in $U$.

Consider now any set of peers $V$, and let $R(v)$ be the subset in $U$ that has been assigned to peer $v$ (for example, by using the hierarchical decomposition approach). Then the following continuous-discrete condition has to be met:

**Condition 13.26.** For every pair of peers $v$ and $w$, $v$ is connected to $w$ if and only if there are two points $x, y \in U$ with $x \in R(v)$, $y \in R(w)$ and $(x, y) \in E_F$.

Let $G_F(V)$ be the graph resulting from this condition. Then the following result holds.

**Lemma 13.27.** If $F$ is mixing and $\cup_v S(v) = U$, then $G_F(V)$ is strongly connected.

**Proof.** Suppose that $F$ is mixing and $\bigcup_{v \in V} R(v) = U$ but $G_F(V)$ is not connected. Then there must be a set $V' \subseteq V$ that has no edge leaving it. Let $R' = \bigcup_{v \in V'} R(v)$ and $R'' = \bigcup_{v \in V \setminus V'} R(v)$. Since $\Gamma(R') \neq \emptyset$ and $\Gamma(R') \subseteq R''$, there must exist an $x \in R'$ and a $y \in R''$ with $\{x, y\} \in E_F$. Hence, according to our definition of $G_F(V)$, there must exist a node $v \in V'$ and a node $w \in V \setminus V'$ with $(v, w) \in E$, contradicting our assumption. \qed
13.2. DECENTRALIZED OVERLAY NETWORKS

Hence, it is important to make sure that $F$ is mixing and that $\cup_v R(v) = U$. The continuous-discrete approach has the following advantage.

**Fact 13.28.** When using the continuous-discrete approach together with consistent hashing or hierarchical decomposition, then for any set of functions $F$ it holds: For each join request of some peer $p$, $p$ only has to contact one old peer (namely, the one containing its region) for a region update and to learn about all of its connections in the system. For each leave request, at most two other peers have to be contacted in order to update their regions.

This ensures that the continuous-discrete approach is highly scalable, as long as $F$ is chosen so that the peers have at most polylogarithmic degree. Next we consider specific examples of decentralized overlay networks based on $U = [0, 1)$. We start with the dynamic hypercube, and then we consider the dynamic de Bruijn network.

13.2.3 The Dynamic Hypercube

Recall the definition of the $d$-dimensional hypercube. Let $V$ be its node set and $E$ be its edge set. All nodes $v \in V$ have labels $(v_1, \ldots, v_d) \in \{0, 1\}^d$, and two nodes $v$ and $w$ are connected if and only if $H(v, w) = 1$. When associating each node $v$ with the point $x_v = \sum_{i=1}^d v_i/2^i \in [0, 1)$ and letting $d \to \infty$, then $V = [0, 1)$ and $E$ is determined by the set $F_H$ of functions $f_i$ for all $i \geq 1$ with $f_i(x) = x \oplus 1/2^i \pmod{1}$ where $\oplus$ is the bit-wise XOR of the binary representations of $x$ and $1/2^i$, i.e., the $i$-th bit in $x$ is reversed. Let $F$ be the set of all functions $f_i^-$ and $f_i^+$ with $f_i^-(x) = x - 1/2^i \pmod{1}$ and $f_i^+(x) = x + 1/2^i \pmod{1}$. Then for every $x \in [0, 1)$, either $f_i^-(x) = f_i(x)$ or $f_i^+(x) = f_i(x)$, so $F$ can be seen as a superset of $F_H$. For simplicity, we will view $([0, 1), E_F)$ as the continuous form of the hypercube. Our choice of $F$ ensures that for each $(x, y) \in E_F$ also $(y, x) \in E_F$.

Consider using the consistent hashing approach in order to partition $[0, 1)$ among the peers. The dynamic hypercube for the decentralized case is based on the continuous-discrete approach together with a cycle connecting each peer to its predecessor and successor in $[0, 1)$. If the peers are assigned to random points in $[0, 1)$, the topological properties of hypercubes together with Lemmas 13.24 and 13.25 imply the following result.

**Lemma 13.29.** Given $n$ peers, the dynamic hypercube has a maximum degree of $O(\log^2 n)$, w.h.p.

Routing in a Dynamic Hypercube

Suppose that we want to route a message from point $x$ to point $y$ in $[0, 1)$. Let $(x_1, x_2, \ldots)$ be the binary representation of $x$ and $(y_1, y_2, \ldots)$ be the binary representation of $y$. Then we use the following continuous strategy to route the message from $x$ to $y$:

$$(x_1, x_2, \ldots) \rightarrow (y_1, x_2, \ldots) \rightarrow (y_1, y_2, x_3, \ldots) \rightarrow \ldots$$

If $x$ and $y$ have infinite binary representations, then this strategy may take an infinite amount of hops, but in the discrete world with a finite number of peers, this is not the case with the following discrete variant of the continuous routing strategy above:

The message starts at the peer $v_0$ responsible for $x$. Peer $v_0$ forwards the message to the peer $v_1$ responsible for $(y_1, x_2, \ldots)$, peer $v_1$ forwards it to the peer $v_2$ responsible
for \((y_1, y_2, x_3, \ldots)\), and so on, until the message reaches a peer \(v_k\) whose region or whose neighboring region contains \(y\). From this peer the message is forwarded to the peer responsible for the region containing \(y\).

Notice that the maximal remaining distance to \(y\) shrinks by a factor 2 in each step. Hence, once a distance equal to the smallest region is reached, the routing terminates. Thus, the following theorem immediately follows from Lemma 13.24.

**Theorem 13.30.** Using the continuous-discrete routing strategy, it takes at most \(O(\log n)\) hops until a message is routed from any point \(x\) to any point \(y\) in \([0, 1)\).

Besides having a small dilation, it is also important to have a small congestion, i.e., when routing multiple messages, the maximum number of messages to be handled by a peer should be as close to optimal as possible. In order to achieve a low congestion, the following routing strategy may be used by any peer, which is a continuous version of Valiant’s trick:

Suppose that a peer with position \(x\) wants to send a message to position \(y\). Then it chooses a random point \(z \in [0, 1)\), first routes the message from \(x\) to \(z\) and then from \(z\) to \(y\) using the continuous-discrete routing strategy above.

With this strategy, we obtain the following theorem.

**Theorem 13.31.** For every permutation routing problem, the congestion caused when using the extended continuous-discrete routing strategy above is at most \(O(\log^2 n)\), w.h.p.

**Proof.** Consider any permutation routing problem \(\pi\), and consider cutting \([0, 1)\) into \(n'\) intervals of size \(1/n'\) starting at integral multiples of \(1/n'\) where \(n'\) is chosen so that \(n'\) is a power of 2 and 
\[
1/n' = (c \ln n)/n
\]
for some suitably chosen constant \(c\). It follows from Lemma 13.25 that every interval has \(O(\log n')\) packets starting at it and \(O(\log n')\) packets aiming for it. Viewing these intervals as the nodes of a \(\log n'\)-dimensional hypercube, it follows from the analysis of Valiant’s trick that at most \(O(\log^2 n')\) packets pass every node, w.h.p. Since, according to Lemma 13.24, every peer is responsible for an interval of size at most \(O(\log n/n)\), this implies that every peer is passed by at most \(O(\log^2 n)\) packets, w.h.p., which proves the theorem.

### Joining and Leaving a Dynamic Hypercube

We only consider isolated executions of join and leave requests because otherwise it can be quite tricky to correctly update the network.

Suppose that a new peer \(v\) contacts some peer \(w\) already in the system to join the system. Then \(v\)’s request is first sent to the peer \(u\) owning \(x_v\) using the continuous-discrete routing strategy, which only takes \(O(\log n)\) hops according to Theorem 13.30. \(u\) forwards information about all of its outgoing edges to \(v\), deletes all edges that it does not need any more, and informs the corresponding endpoints about this. Because \(R(v) \subseteq R(u)\) for the old \(R(u)\), the edges reported to \(v\) are a superset of the edges that it needs to establish. \(v\) checks which of the edges are relevant for it, informs the other endpoint for each relevant edge, and removes the others.

If a peer \(v\) wants to leave the network, it simply forwards all of its outgoing edges to the peer at \(\text{pred}(x_v)\). That peer will then merge these edges with its existing edges and notify the endpoints of these edges about the changes.

We know from Theorem 13.30 that the routing part only takes \(O(\log n)\) hops. Furthermore, Lemmas 13.24 and 13.25 imply that every peer has at most \(O(\log^2 n)\) incoming and outgoing edges. Hence, we obtain the following theorem.
Theorem 13.32. Join and leave require at most $O(\log^2 n)$ work and $O(\log n)$ communication rounds, w.h.p.

Data management
Suppose that we want to store data in the dynamic hypercube. Here we can simply use the consistent hashing strategy in Section 4: data items are hashed to random values in $[0, 1)$ using a pseudo-random hash function $h$ (which is known to all peers), and every data item $d$ is stored in the peer $v$ with $h(d) \in R_v$.

Using this strategy, data will, on expectation, be evenly distributed among the peers, and on expectation, at most a factor of 2 more data than necessary has to be replaced if a node joins or leaves. (Recall the section on hashing.)

13.2.4 The Dynamic de Bruijn Network
Recall the definition of the continuous de Bruijn graph. According to this definition,
- $U = [0, 1)$ and
- $F = \{f_0, f_1\}$ with $f_0(x) = x/2$ and $f_1(x) = (1 + x)/2$.

The dynamic de Bruijn graph for the decentralized case is based on the continuous-discrete approach together with a cycle connecting each peer to its predecessor and successor in $[0, 1)$. It follows from Lemmas 13.24 and 13.25:

Lemma 13.33. Given $n$ peers, the dynamic de Bruijn network has a maximum degree of $O(\log n)$ and a diameter of $O(\log n)$, w.h.p.

Routing in a Dynamic de Bruijn Network
This is done in the same way as described for the supervised de Bruijn graph.

Joining and Leaving a Dynamic de Bruijn Network
Suppose that a new peer $v$ contacts some peer $w$ already in the system to join the system. Then $v$’s request is first sent to the peer $u$ owning $x_v$ using the continuous-discrete routing strategy above, which only takes $O(\log n)$ hops according to Section 6.6. $u$ forwards information about all of its (incoming and) outgoing edges to $v$, deletes all edges that it does not need any more, and informs the corresponding endpoints about this. Because $R(v) \subseteq R(u)$ for the old $R(u)$, the edges reported to $v$ are a superset of the edges that it needs to establish. $v$ checks which of the edges are relevant for it, informs the other endpoint for each relevant edge, and removes the others.

If a node $v$ wants to leave the network, it simply forwards all of its outgoing edges to the peer at $\text{pred}(x_v)$. That peer will then merge these edges with its existing edges and notifies the endpoints of these edges about the changes.

We know from Section 6.6 that the routing part only takes $O(\log n)$ hops. Furthermore, Lemmas 13.24 and 13.25 imply that every peer has at most $O(\log n)$ outgoing edges. Hence, we obtain the following theorem.

Theorem 13.34. Join and leave take at most $O(\log n)$ work and $O(\log n)$ communication rounds, w.h.p.

Also the dynamic de Bruijn network can be used for data management with the help of the consistent hashing approach.
Dynamic Gabber-Galil Graph

Recall the Gabber-Galil graph with parameter $n$. For this graph, $V = [n]^2$ and $E$ consists of all edges $\{(x, y), (x', y')\}$ with

$$(x', y') \in \{(x, x + y), (x, x + y + 1), (x + y, y), (x + y + 1, y)\} \pmod{n}$$

When transforming $V$ into $\{i/n \mid i \in \{0, \ldots, n - 1\}\}^2$ and letting $n \to \infty$, then we obtain an node set $V = [0, 1)^2$ and an edge set $E_F$ specified by the functions $f_1(x, y) = (x, x + y) \pmod{1}$, $f_2(x, y) = (x + y, y) \pmod{1}$, $f_3(x, y) = f_1^{-1}(x, y) = (x, y - x) \pmod{1}$ and $f_4(x, y) = f_2^{-1}(x, y) = (x - y, y) \pmod{1}$. Thus, $([0, 1)^2, E_F)$ can be seen as a continuous version of the Gabber-Galil graph. One can show the following result:

**Theorem 13.35.** Suppose that we have $n$ peers. When using random labels together with the hierarchical decomposition to assign each peer $v$ to a subcube in $[0, 1)^2$, the graph $G_F(V)$ has a degree of $O(\log n)$, diameter of $O(\log n)$ and expansion of $\Omega(1/\log n)$, with high probability.

Routing in the dynamic form of the Gabber-Galil graph is very difficult since expanders tend to have a very irregular structure. So we do not describe how to do that here. Joining and leaving is done following the hierarchical decomposition approach. Since whenever a peer joins, a subcube is split into two, a new peer $v$ can get all of its connections from the peer $u$ whose subcube is split. Whenever a peer $v$ leaves, we either merge to subcubes or take one peer $w$ to take over the subcube of $v$ and merge the subcubes of $w$ and its sibling $w'$ in the decomposition tree into one that is assigned to $w'$. In any case, we obtain the following result.

**Theorem 13.36.** It takes a routing effort of $O(\log n)$ hops and an update work of $O(\log n)$ messages that can be processed in a logarithmic number of communication rounds in order to execute a join or leave operation in the dynamic Gabber-Galil graph.

### 13.2.5 Robustness Against Random Faults

In order to protect against random faults in dynamic networks based on $U = [0, 1)$, each peer $v$ aims at preserving the following condition (which is similar to the supervised case).

**Condition 13.37.** Every peer $v$ in the system is connected to

- pred$_i(v)$ and succ$_i(v)$ for every $i \in \{1, \ldots, k\}$ and

- all peers $w$ with the property that $\Gamma(R(N_w)) \cap R(N_v) \neq \emptyset$

where $N_v = \{v\} \cup \{\text{pred}_i(v) \mid i \in \{1, \ldots, k\}\} \cup \{\text{succ}_i(v) \mid i \in \{1, \ldots, k\}\}$ and for any set $V' \subseteq V$, $R(V') = \bigcup_{v \in V'} R(v)$.

When doing this, the following result can be shown.

**Theorem 13.38.** If faults of peers are independent of their positions and only happen at a (sufficiently small) constant rate, then the peers can maintain Condition 13.37 everywhere, w.h.p.
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The result holds since for a sufficiently small constant fault rate and a sufficiently large $k$, no entire sequence of $k$ consecutive peers in $[0, 1)$ will become faulty within a constant number of steps, w.h.p., and every peer only needs a constant number of communication rounds to update its connectivity information as peers among its predecessors and successors become faulty as long as this is in principle possible (i.e., not all of its predecessors and successors fail).

13.2.6 Robustness Against Adversarial Join-Leave Behavior

Finally, we consider the problem of protecting an overlay network against adversarial join-leave behavior. More precisely, we consider the following scenario. There are $n$ blue (or honest) nodes and $\epsilon n$ red (or adversarial) nodes for some fixed constant $\epsilon < 1$. There is a rejoin operation that, when applied to node $v$, lets $v$ first leave the system and then join it again from scratch. The leaving is done by simply removing $v$ from the system and the joining is done with the help of a join operation to be specified by the system. We assume that the sequence of rejoin requests is controlled by an adversary, which is a typical assumption in the analysis of online algorithms. The adversary can only issue rejoin requests for the red nodes, but it can do this in an arbitrary adaptive manner. That is, at any time it can inspect the entire system and select whatever red node it likes to rejoin the system. Our goal is to find an oblivious join strategy, i.e., a strategy that cannot distinguish between the blue and red nodes, so that for any adversarial strategy above the following two conditions can be preserved for every interval $I \subseteq [0, 1)$ of size at least $(c \log n)/n$ for a constant $c > 0$ and any polynomial number of rounds in $n$:

- **Balancing condition**: $I$ contains $\Theta(|I| \cdot n)$ nodes.
- **Majority condition**: the blue nodes in $I$ are in the majority.

It is not difficult to see that the brute-force strategy of giving every node a new random place whenever a node rejoins will achieve the stated goal, with high probability, but this would be a very expensive strategy. The challenge is to find a join operation that needs as little randomness and as few rearrangements as possible to satisfy the two conditions. Fortunately, there is such a strategy, called the **cuckoo rule**. We first introduce some notation, and then we describe the strategy.

In the following, a **region** is an interval of size $1/2^r$ in $[0, 1)$ for some integer $r$ that starts at an integer multiple of $1/2^r$. Hence, there are exactly $2^r$ regions of size $1/2^r$. A **$k$-region** is a region of size (closest from above to) $k/n$, and for any point $x \in [0, 1)$, the $k$-region $R_k(x)$ is the unique $k$-region containing $x$.

**Cuckoo rule**: If a new node $v$ wants to join the system, pick a random $x \in [0, 1)$. Place $v$ into $x$ and move all nodes in $R_k(x)$ to points in $[0, 1)$ chosen uniformly and independently at random (without replacing any further nodes).

See Figure 13.7 for an illustration of the cuckoo rule. The following result can be shown:

**Theorem 13.39.** For any constants $\epsilon$ and $k$ with $\epsilon < 1 - 1/k$, the cuckoo rule with parameter $k$ satisfies the balancing and majority conditions for a polynomial number of rounds, with high probability, for any adversarial strategy within our model. The inequality $\epsilon < 1 - 1/k$ is sharp as counterexamples can be constructed otherwise.
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Figure 13.7: The cuckoo rule. (a) A new node is placed at a random point $x$. (b) Old nodes in $R_k(x)$ are evicted and moved to new, random places.

We just sketch the proof of the theorem. Let $\hat{R}$ be any fixed region of size $(c \log n) \cdot k/n$, for some constant $c$, for which we want to check the balancing and majority conditions over polynomial in $n$ many steps. Thus, $\hat{R}$ contains exactly $c \log n$ many $k$-regions. The age of a $k$-region is the difference between the current round and the last round when a new node was placed into it (and all old nodes got evicted), and the age of $\hat{R}$ is defined as the sum of the ages of its $k$-regions. A node in $\hat{R}$ is called new if it was placed in $\hat{R}$ when it joined the system, and otherwise it is called old. The first lemma follows directly from the cuckoo rule because every $k$-region can have at most one new node at any time.

**Lemma 13.40.** At any time, $\hat{R}$ contains at most $c \log n$ new nodes.

In order to bound the number of old nodes in $\hat{R}$, we first have to bound the age of $\hat{R}$ (Lemma 13.41). Then we bound the maximum number of nodes in a $k$-region (Lemma 13.42) and use this to bound the number of evicted blue and red nodes in a certain time interval (Lemma 13.43). After that, we can combine all lemmas to bound the number of old blue and red nodes in $\hat{R}$ (Lemma 13.44).

**Lemma 13.41.** At any time, $\hat{R}$ has an age within $(1 \pm \delta)(c \log n) n/k$, with high probability, where $\delta > 0$ is a constant that can be made arbitrarily small depending on the constant $c$.

**Lemma 13.42.** For any $k$-region $R$ in $\hat{R}$ it holds at any time that $R$ has at most $O(k \log n)$ nodes, with high probability.

Next we bound the number of blue and red nodes that are evicted in a certain time interval.

**Lemma 13.43.** For any time interval $I$ of size $T = (\gamma/\epsilon) \log^3 n$, the number of blue nodes that are evicted in $I$ is within $(1 \pm \delta)T \cdot k$, with high probability, and the number of red nodes that are evicted in $I$ is within $(1 \pm \delta)T \cdot \epsilon k$, with high probability, where $\delta > 0$ can be made arbitrarily small depending on $\gamma$.

Combining Lemmas 13.41 to 13.43, we obtain the following lemma.

**Lemma 13.44.** At any time, $\hat{R}$ has within $(1 \pm \delta)(c \log n) \cdot k$ old blue nodes and within $(1 \pm \delta)(c \log n) \cdot \epsilon k$ old red nodes, with high probability, where the lower bound on the red nodes holds if none of the red nodes has rejoined.

Combining Lemmas 13.40 and 13.44, we can now prove when the balancing and majority conditions are satisfied.
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- **Balancing condition:** From Lemmas 13.40 and 13.44 it follows that every region $R$ of size $(c \log n)k/n$ has at least $(1 - \delta)(c \log n) \cdot k$ and at most $(1 + \delta)(c \log n + (c \log n)k + (c \log n)k) = (1 + \delta)(c \log n)(1 + (1 + \epsilon)k)$ nodes, where the constant $\delta > 0$ can be made arbitrarily small. Hence, the regions are balanced within a factor of close to $(1 + \epsilon + 1/k)$.

- **Majority condition:** From Lemmas 13.40 and 13.44 it also follows that every region of size $(c \log n)k/n$ has at least $(1 - \delta)(c \log n) \cdot k$ blue nodes and at most $(1 + \delta)(c \log n + (c \log n) \cdot k)$ red nodes, w.h.p., where the constant $\delta > 0$ can be made arbitrarily small. These bounds are also tight in the worst case, which happens if the adversary focuses on a specific region $R$ of size $(c \log n)k/n$ and continuously rejoins with any red node outside of $R$. Hence, the adversary is not able to obtain the majority in any region of size $(c \log n)k/n$ as long as $(c \log n)(k + 1) < (c \log n) \cdot k$ which is true if and only if $\epsilon < 1 - 1/k$.

Hence, for $\epsilon < 1 - 1/k$ the balancing and majority conditions are satisfied, w.h.p., and this is sharp, which proves Theorem 13.39.

The cuckoo rule has the drawback that it only works if only the red nodes show adversarial join-leave behavior. What if both kinds of nodes show adversarial join-leave behavior? Then we need to extend the cuckoo rule in the following way in order to maintain the balancing and majority conditions.

The join operation works in the same way as the cuckoo rule. But whenever a peer wants to leave the network, we use the following leave operation:

**Leave($v$):** If a peer $v$ leaves the system, then a $k$-region $R$ is chosen uniformly at random among the $k$-regions of $R_{c \log n}(x)$ for some (sufficiently large) constant $c$, where $x$ is the position of $v$. $R$ is flipped with a $k$-region $R'$ chosen uniformly at random in $[0, 1)$, and then all peers in $R$ (as well as $v$) have to rejoin the system from scratch using the cuckoo rule.

Hence, the departure of a peer may spawn several join operations. We call this algorithm the **cuckoo&flip strategy**. With this strategy the balancing and majority conditions can be kept, with high probability. More precisely, one can show the following result:

**Theorem 13.45.** For any constants $\epsilon$ and $k$ with $\epsilon < 1/4 - (2 \log k + 1)/k$, the cuckoo&flip strategy satisfies the balancing and majority conditions for any polynomial number of rejoin requests, with high probability, for any adversarial strategy within our model.

Hence, as long as $\epsilon < 1/4$, only a constant factor overhead has to be paid (on average) compared to standard join and leave operations without any additional replacements of peers.