Simple CHT:
A New Derivation of the Weakest Failure Detector for Consensus

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Abstract

The paper proposes an alternative proof that \( \Omega \), an oracle that outputs a process identifier and guarantees that eventually the same correct process identifier is output at all correct processes, provides minimal information about failures for solving consensus in read-write shared-memory systems: every oracle that gives enough failure information to solve consensus can be used to implement \( \Omega \).

Unlike the original proof by Chandra, Hadzilacos and Toueg (CHT), the proof presented in this paper builds upon the very fact that 2-process wait-free consensus is impossible. Also, since the oracle that is used to implement \( \Omega \) can solve consensus, the implementation is allowed to directly access consensus objects. As a result, the proposed proof is shorter and conceptually simpler than the original one.

1 Introduction

The presence of faults and the lack of synchrony make distributed computing interesting. In an asynchronous system which assumes no bounds on communication delays and relative processing speeds, even a basic form of non-trivial synchronization (consensus) is impossible if just one process may fail by crashing [7, 13]. On the other hand, in a synchronous system, in the bounds exist and known a priori, almost any meaningful fault-tolerant synchronization problem becomes solvable. On the other hand, This gap suggests that the amount of synchrony is a crucial factor in reasoning about solvability of distributed computing problems in the presence of faults.

Chandra and Toueg proposed failure detectors as a convenient language to describe synchrony assumptions. Informally, a failure detector is a distributed oracle that provides processes with hints about failures [5]. The notion of a weakest failure detector [4] captures the exact amount of synchrony needed to solve a given problem: \( D \) is the weakest failure detector for solving \( M \) if (1) \( D \) is sufficient to solve \( M \), i.e., there exists an algorithm that solves \( M \) using \( D \), and (2) any failure detector \( D' \) that is sufficient to solve \( M \) provides at least as much information about failures as \( D \) does, i.e., there exists a reduction algorithm that extract the output of \( D \) using the failure information provided by \( D' \).

This paper considers a distributed system in which \( n \) crash-prone processes communicate using atomic reads and writes in shared memory. In the (binary) consensus problem [7], every process starts with a binary input and every correct (never-failing) process is supposed to output one of the inputs such that no two processes output different values. Asynchronous (wait-free) consensus is known to be impossible [7, 13], as long as at least one process may fail by crashing. Chandra et al. [4] showed that the “eventual leader” failure
The reduction algorithm. Section 4 overviews the related work and Section 5 concludes the paper by dis-

access consensus objects, since it is given a failure detector which can be used to solve consensus. The observation here is that a reduction algorithm may directly always output the same correct process. The rest of the paper is organized as follows. Section 2 describes the system model. Sections 3 presents the reduction algorithm. Section 4 overviews the related work and Section 5 concludes the paper by dis-
cussing implications of the presented results.

2 Model

The model of processes communicating through read-write shared objects and using failure detectors is based on [4, 9, 10, 11]. The details necessary for showing the results of this paper are described below.

2.1 Processes and objects

A distributed system is composed of a set \( \Pi \) of \( n \) processes \( \{p_1, \ldots, p_n\} \) \((n \geq 2)\). Processes are subject to crash failures. A process that never fails is said to be correct. Processes that are not correct are called faulty. Process communicate through applying atomic operations on a collection of shared objects. In this paper, the shared objects are registers, i.e., they export only conventional atomic read-write operations.

2.2 Failure patterns and failure detectors

A failure pattern \( F \) is a function from the time range \( \mathbb{T} = \{0\} \cup \mathbb{N} \) to \( 2^\Pi \), where \( F(t) \) denotes the set of processes that have crashed by time \( t \). Once a process crashes, it does not recover, i.e., \( \forall t : F(t) \subseteq F(t+1) \). The set of faulty processes in \( F \), \( \cup_{t \in \mathbb{T}} F(t) \), is denoted by faulty(\( F \)). Respectively, correct(\( F \)) = \( \Pi \) - faulty(\( F \)). A process \( p \in F(t) \) is said to be crashed at time \( t \). An environment is a set of failure patterns. This paper considers environments that consists of failure patterns in which at least one process is correct.

A failure detector history \( H \) with range \( \mathcal{R} \) is a function from \( \Pi \times \mathbb{T} \) to \( \mathcal{R} \). \( H(p_i, t) \) is interpreted as the value output by the failure detector module of process \( p_i \) at time \( t \). A failure detector \( \mathcal{D} \) with range \( \mathcal{R}_\mathcal{D} \) is a function that maps each failure pattern to a (non-empty) set of failure detector histories with range \( \mathcal{R}_\mathcal{D} \). \( \mathcal{D}(F) \) denotes the set of possible failure detector histories permitted by \( \mathcal{D} \) for failure pattern \( F \). Possible ranges of failure detectors are not a priori restricted.

2.3 Algorithms

An algorithm \( \mathcal{A} using a failure detector \mathcal{D} \) is a collection of deterministic automata, one for each process in the system. \( \mathcal{A}_i \) denotes the automaton on which process \( p_i \) runs the algorithm \( \mathcal{A} \). Computation proceeds in atomic steps of \( \mathcal{A} \). In each step of \( \mathcal{A} \), process \( p_i \)

(i) invokes an atomic operation (read or write) on a shared object and receives a response or queries its failure detector module \( \mathcal{D}_i \) and receives a value from \( \mathcal{D} \), and

(ii) applies its current state, the response received from the shared object or the value output by \( \mathcal{D} \) to the automaton \( \mathcal{A}_i \) to obtain a new state.

A step of \( \mathcal{A} \) is thus identified by a tuple \( (p_i, d) \), where \( d \) is the failure detector value output at \( p_i \) during that step if \( \mathcal{D} \) was queried, and \( \perp \) otherwise.

If the state transitions of the automata \( \mathcal{A}_i \) do not depend on the failure detector values, the algorithm \( \mathcal{A} \) is called asynchronous. Thus, for an asynchronous algorithm, a step is uniquely identified by the process id.
2.4 Runs

A state of $A$ defines the state of each process and each object in the system. An initial state $I$ of $A$ specifies an initial state for every automaton $A_i$ and every shared object.

A run of algorithm $A$ using a failure detector $D$ in an environment $E$ is a tuple $R = \langle F, H, I, S, T \rangle$ where $F \in E$ is a failure pattern, $H \in D(F)$ is a failure detector history, $I$ is an initial state of $A$, $S$ is an infinite sequence of steps of $A$ respecting the automata $A$ and the sequential specification of shared objects, and $T$ is an infinite list of increasing time values indicating when each step of $S$ has occurred, such that for all $k \in \mathbb{N}$, if $S[k] = (p_i, d)$ with $d \neq \bot$, then $p_i \notin F(T[k])$ and $d = H(p_i, T[k])$.

A run $\langle F, H, I, S, T \rangle$ is fair if every process in $\text{correct}(F)$ takes infinitely many steps in $S$, and $k$-resilient if at least $n - k$ processes appear in $S$ infinitely often. A partial run of an algorithm $A$ is a finite prefix of a run of $A$.

For two steps $s$ and $s'$ of processes $p_i$ and $p_j$, respectively, in a (partial) run $R$ of an algorithm $A$, we say that $s$ causally precedes $s'$ if in $R$, and we write $s \rightarrow s'$, if (1) $p_i = p_j$, and $s$ occurs before $s'$ in $R$, or (2) $s$ is a write step, $s'$ is a read step, and $s$ occurs before $s'$ in $R$, or (3) there exists $s''$ in $R$, such that $s \rightarrow s''$ and $s'' \rightarrow s'$.

2.5 Consensus

In the binary consensus problem, every process starts the computation with an input value in $\{0, 1\}$ (we say the process proposes the value), and eventually reaches a distinct state associated with an output value in $\{0, 1\}$ (we say the process decides the value). An algorithm $A$ solves consensus in an environment $E$ if in every fair run of $A$ in $E$, (i) every correct process eventually decides, (ii) every decided value was previously proposed, and (iii) no two processes decide different values.

Given an algorithm that solves consensus, it is straightforward to implement an abstraction $\text{CONS}$ that can be accessed with an operation $\text{propose}(v)$ ($v \in \{0, 1\}$) returning a value in $\{0, 1\}$, and guarantees that every $\text{propose}$ operation invoked by a correct process eventually returns, every returned value was previously proposed, and no two different values are ever returned.

2.6 Weakest failure detector

We say that an algorithm $A$ using $D'$ extracts the output of $D$ in $E$, if $A$ implements a distributed variable $D$-output such that for every run $R = \langle F, H', S, T \rangle$ of $A$ in which $F \in E$, there exists $H \in D(F)$ such that for all $p_i \in \Pi$ and $t \in T$, $D$-output$_i(t) = H(p_i, t)$ (i.e., the value of $D$-output at $p_i$ at time $t$ is $H(p_i, t)$). We say that $A$ is a reduction algorithm. (A more precise definition of a reduction algorithm is given in [11].)

If, for failure detectors $D$ and $D'$ and an environment $E$, there is a reduction algorithm using $D'$ that extracts the output $D$ in $E$, then we say that $D$ is weaker than $D'$ in $E$.

$D$ is the weakest failure detector to solve a problem $M$ (e.g., consensus) in $E$ if there is an algorithm that solves $M$ using $D$ in $E$ and $D$ is weaker than any failure detector that can be used to solve $M$ in $E$.

3 Extracting $\Omega$

Let $A$ be an algorithm that solves consensus using a failure detector $D$. The goal is to construct an algorithm that emulates $\Omega$ using $A$ and $D$. Recall that to emulate $\Omega$ means to output, at each time and at each process, a process identifiers such that, eventually, the same correct process is always output.
Shared variables:
for all \( p_i \in \Pi \): \( G_i \), initially empty graph

\begin{verbatim}
1  \( k_i := 0 \)
2  \textbf{while true do}
3  \textbf{for all } \( p_j \in \Pi \) \textbf{do} \( G_i \leftarrow G_i \cup G_j \)
4  \( d_i := \text{query failure detector} \ D \)
5  \( k_i := k_i + 1 \)
6  \text{add } [p_i, d_i, k_i] \text{ and edges from all other vertices}
   \text{of } G_i \text{ to } [p_i, d_i, k_i], \text{ to } G_i
\end{verbatim}

**Figure 1.** Building a DAG: the code for each process \( p_i \)

### 3.1 Overview

As in [4], the reduction algorithm of this paper uses failure detector \( D \) to construct an ever-growing *directed acyclic graph* (DAG) that contains a sample of the values output by \( D \) in the current run and captures some temporal relations between them. Following [15], this DAG can be used by an *asynchronous* algorithm \( A' \) to simulate a (possibly finite and unfair) run of \( A \). Recall that, using BG-simulation [1, 3], \( 1 \)-resilient runs of \( A' \) can be simulated. The fact that \( 1 \)-resilient 2-process consensus is impossible implies that the simulation must produce at least one "non-deciding" \( 1 \)-resilient run of \( A' \).

Now every correct process locally simulates all executions of BG-simulation on two processes \( q_1 \) and \( q_2 \) that simulate a \( 1 \)-resilient run of \( A' \) of the whole system \( \Pi \). Eventually, every correct process locates a never-deciding run of \( A' \) and uses the run to extract the output of \( \Omega \): it is sufficient to output the process that takes the least number of steps in the "smallest" non-deciding simulated run of \( A' \). Indeed, exactly one correct process takes finitely many steps in the non-deciding \( 1 \)-resilient run of \( A' \): otherwise, the run would simulate a fair and thus deciding run of \( A \).

The reduction algorithm extracting \( \Omega \) from \( A \) and \( D \) consists of two components that are running in parallel: the *communication component* and the *computation component*. In the communication component, every process \( p_i \) maintains the ever-growing directed acyclic graph (DAG) \( G_i \) by periodically querying its failure detector module and exchanging the results with the others through the shared memory. In the computation component, every process simulates a set of runs of \( A \) using the DAGs and maintains the extracted output of \( \Omega \).

### 3.2 DAGs

The communication component is presented in Figure 1. This task maintains an ever-growing DAG that contains a finite sample of the current failure detector history. The DAG is stored in a register \( G_i \) which can be updated by \( p_i \) and read by all processes.

DAG \( G_i \) has some special properties which follow from its construction [4]. Let \( F \) be the current failure pattern, and \( H \in D(F) \) be the current failure detector history. Then a fair run of the algorithm in Figure 1 guarantees that there exists a map \( \tau : \Pi \times \mathcal{R}_D \times \mathbb{N} \rightarrow \mathbb{T} \), such that, for every correct process \( p_i \) and every time \( t \) (\( x(t) \) denotes here the value of variable \( x \) at time \( t \)):

1. The vertices of \( G_i(t) \) are of the form \([p_j, d, \ell]\) where \( p_j \in \Pi \), \( d \in \mathcal{R}_D \) and \( \ell \in \mathbb{N} \).
(a) For each vertex \( v = [p_j, d, \ell], p_j \notin F(\tau(v)) \) and \( d = H(p_j, \tau(v)) \). That is, \( d \) is the value output by \( p_j \)'s failure detector module at time \( \tau(v) \).

(b) For each edge \((v, v')\), \( \tau(v) < \tau(v') \). That is, any edge in \( G_i \) reflects the temporal order in which the failure detector values are output.

2. If \( v = [p_j, d, \ell] \) and \( v' = [p_j, d', \ell'] \) are vertices of \( G_i(t) \) and \( \ell < \ell' \) then \((v, v')\) is an edge of \( G_i(t) \).

3. \( G_i(t) \) is transitively closed: if \((v, v')\) and \((v', v'')\) are edges of \( G_i(t) \), then \((v, v'')\) is also an edge of \( G_i(t) \).

4. For all correct processes \( p_j \), there is a time \( t' \geq t \), a \( d \in \mathcal{D} \) and a \( \ell \in \mathbb{N} \) such that, for every vertex \( v \) of \( G_i(t) \), \((v, [p_j, d, \ell])\) is an edge of \( G_i(t') \).

5. For all correct processes \( p_j \), there is a time \( t' \geq t \) such that \( G_i(t) \) is a subgraph of \( G_j(t') \).

The properties imply that ever-growing DAGs at correct processes tend to the same infinite DAG \( G \): \( \lim_{t \to \infty} G_i(t) = G \). In a fair run of the algorithm in Figure 1, the set of processes that obtain infinitely many vertices in \( G \) is the set of correct processes [4].

3.3 Asynchronous simulation

It is shown below that any infinite DAG \( G \) constructed as shown in Figure 1 can be used to simulate partial runs of \( \mathcal{A} \) in the asynchronous manner: instead of querying \( \mathcal{D} \), the simulation algorithm \( \mathcal{A}' \) uses the samples of the failure detector output captured in the DAG. The pseudo-code of this simulation is presented in Figure 2. The algorithm is hypothetical in the sense that it uses an infinite input, but this requirement is relaxed later.

In the algorithm, each process \( p_i \) is initially associated with an initial state of \( \mathcal{A} \) and performs a sequence of simulated steps of \( \mathcal{A} \). Every process \( p_i \) maintains a shared register \( V_i \) that stores the vertex of \( G \) used for the most recent step of \( \mathcal{A} \) simulated by \( p_i \). Each time \( p_i \) is about to perform a step of \( \mathcal{A} \) it first reads registers \( V_1, \ldots, V_n \) to obtain the vertexes of \( G \) used by processes \( p_1, \ldots, p_n \) for simulating the most recent causally preceding steps of \( \mathcal{A} \) (line 10 in Figure 2). Then \( p_i \) selects the next vertex of \( G \) that succeeds all vertices (lines 11-14). If no such vertex is found, \( p_i \) blocks forever (line 13).

Note that a correct process \( p_i \) may block forever if \( G \) contains only finitely many vertices of \( p_i \). As a result an infinite run of \( \mathcal{A}' \) may simulate an unfair run of \( \mathcal{A} \): a run in which some correct process takes only finitely many steps. But every finite run simulated by \( \mathcal{A}' \) is a partial run of \( \mathcal{A} \).

**Theorem 1** Let \( G \) be the DAG produced in a fair run \( R = \langle F, H, I, S, T \rangle \) of the communication component in Figure 1. Let \( R' = \langle F', H', I', S', T' \rangle \) be any fair run of \( \mathcal{A}' \) using \( G \). Then the sequence of steps simulated by \( \mathcal{A}' \) in \( R' \) belongs to a (possibly unfair) run of \( \mathcal{A} \), \( R_{\mathcal{A}} \), with input vector of \( I' \) and failure pattern \( F \). Moreover, the set of processes that take infinitely many steps in \( R_{\mathcal{A}} \) is \( \text{correct}(F) \cap \text{correct}(F') \), and if \( \text{correct}(F) \subseteq \text{correct}(F') \), then \( R_{\mathcal{A}} \) is fair.

**Proof.** Recall that a step of a process \( p_i \) can be either a memory step in which \( p_i \) accesses shared memory or a query step in which \( p_i \) queries the failure detector. Since memory steps simulated in \( \mathcal{A}' \) are performed as in \( \mathcal{A} \), to show that algorithm \( \mathcal{A}' \) indeed simulates a run of \( \mathcal{A} \) with failure pattern \( F \), it is enough to make sure that the sequence of simulated query steps in the simulated run (using vertexes of \( G \)) could have been observed in a run \( R_{\mathcal{A}} \) of \( \mathcal{A} \) with failure pattern \( F \) and the input vector based on \( I' \).
Shared variables:
\[ V_1, \ldots, V_n := \perp, \ldots, \perp, \]
\{for each \( p_i \), \( V_j \) is the vertex of \( G \)
\corresponding to the latest simulated step of \( p_j \)\}

Shared variables of \( A \)

7 initialize the simulated state of \( p_i \) in \( A \), based on \( I' \)
8 \( \ell := 0 \)
9 \textbf{while true do}
10 \{Simulating the next \( p_i \)'s step of \( A \)\}
11 \( U := [V_1, \ldots, V_n] \)
12 \textbf{repeat}
13 \( \ell := \ell + 1 \)
14 \textbf{wait until} \( G \) includes \([p_i, d, \ell]\) for some \( d \)
15 \textbf{until} \( \forall j, U[j] \neq \perp: (U[j], [p_i, d, \ell]) \in G \)
16 \( V_i := [p_i, d, \ell] \)
17 \textbf{take the next} \( p_i \)'s step of \( A \) using \( d \) as the output of \( D \)

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**Figure 2.** DAG-based asynchronous algorithm \( A' \): code for each \( p_i \)

Let \( \tau \) be a map associated with \( G \) that carries each vertex of \( G \) to an element in \( T \) such that (a) for any vertex \( v = [p, d, \ell] \) of \( G \), \( p \notin F(\tau(v)) \) and \( d = H(p_j, \tau(v)) \), and (b) for every edge \((v, v')\) of \( G \), \( \tau(v) < \tau(v') \) (the existence of \( \tau \) is established by property (5) of DAGs in Section 3.2). For each step \( s \) simulated by \( A' \) in \( R' \), let \( \tau'(s) \) denote time when step \( s \) occurred in \( R' \), i.e., when the corresponding line 16 in Figure 2 was executed, and \( v(s) \) be the vertex of \( G \) used for simulating \( s \), i.e., the value of \( V_i \) when \( p_i \) simulates \( s \) in line 16 of Figure 2.

Consider query steps \( s_i \) and \( s_j \) simulated by processes \( p_i \) and \( p_j \), respectively. Let \( v(s_i) = [p_i, d_i, \ell] \) and \( v(s_j) = [p_j, d_j, m] \). WLOG, suppose that \( \tau([p_i, d_i, \ell]) < \tau([p_j, d_j, m]) \), i.e., \( D \) outputs \( d_i \) at \( p_i \) before outputting \( d_j \) at \( p_j \).

If \( \tau'(s_i) < \tau'(s_j) \), i.e., \( s_i \) is simulated by \( p_i \) before \( s_j \) is simulated by \( p_j \), then the order in which \( s_i \) and \( s_j \) see value \( d_i \) and \( d_j \) is the run produced by \( A' \) is consistent with the output of \( D \), i.e., the values \( d_i \) and \( d_j \) indeed could have been observed in that order.

Suppose now that \( \tau'(s_i) > \tau'(s_j) \). If \( s_i \) and \( s_j \) are not causally related in the simulated run, then \( R' \) is indistinguishable from a run in which \( s_i \) is simulated by \( p_i \) before \( s_j \) is simulated by \( p_j \). Thus, \( s_i \) and \( s_j \) can still be observed in a run of \( A \).

Now suppose, by contradiction that \( \tau'(s_i) > \tau'(s_j) \) and \( s_j \) causally precedes \( s_i \) in the simulated run, i.e., \( p_j \) simulated at least one write step \( s'_j \) after \( s_j \), and \( p_i \) simulated at least one read step \( s'_i \) before \( s_i \), such that \( s'_i \) took place before \( s'_j \) in \( R' \). Since before performing the memory access of \( s'_i \), \( p_j \) updated \( V_j \) with a vertex \( v(s'_j) \) that succeeds \( v(s_j) \) in \( G \) (line 15), and \( s'_i \) occurs in \( R' \) after \( s'_j \), \( p_i \) must have found \( v(s'_j) \) or a later vertex of \( p_j \) in \( V_j \) before simulating step \( s_i \) (line 10) and, thus, the vertex of \( G \) used for simulating \( s_i \) must be a descendant of \([p_j, d_j, m]\), and, by properties (1) and (3) of DAGs (Section 3.2), \( \tau([p_i, d_i, \ell]) > \tau([p_j, d_j, m]) \) — a contradiction. Hence, the sequence of steps of \( A \) simulated in \( R' \) could have been observed in a run \( R_A \) of \( A \) with failure pattern \( F \).

Since in \( A' \), a process simulates only its own steps of \( A \), every process that appears infinitely often in \( R_A \) is in \( \text{correct}(F') \). Also, since each faulty in \( F \) process contains only finitely many vertices in \( G \), eventually, each process in \( \text{correct}(F') - \text{correct}(F) \) is blocked in line 13 in Figure 2, and, thus, every process that
appears infinitely often in $R_A$ is also in $\text{correct}(F)$. Now consider a process $p_i \in \text{correct}(F') \cap \text{correct}(F)$. Property (4) of DAGs implies that for every set $V$ of vertices of $G$, there exists a vertex of $p_i$ in $G$ such that for all $v' \in V$, $(v', v)$ is an edge in $G$. Thus, the wait statement in line 13 cannot block $p_i$ forever, and $p_i$ takes infinitely many steps in $R_A$.

Hence, the set of processes that appear infinitely often in $R_A$ is exactly $\text{correct}(F') \cap \text{correct}(F)$. Specifically, if $\text{correct}(F) \subseteq \text{correct}(F')$, then the set of processes that appear infinitely often in $R_A$ is $\text{correct}(F)$, and the run is fair. □

Note that in a fair run, the properties of the algorithm in Figure 2 remain the same if the infinite DAG $G$ is replaced with a finite ever-growing DAG $\bar{G}$ constructed in parallel (Figure 1) such that $\lim_{t \to \infty} \bar{G} = G$. This is because such a replacement only affects the wait statement in line 13 which blocks $p_i$ until the first vertex of $p_i$ that causally succeeds every simulated step recently "witnessed" by $p_i$ is found in $G$, but this cannot take forever if $p_i$ is correct (properties (4) and (5) of DAGs in Section 3.2). The wait blocks forever if the vertex is absent in $G$, which may happen only if $p_i$ is faulty.

3.4 BG-simulation

Borowsky and Gafni proposed in [1, 3], a simulation technique by which $k + 1$ simulators $q_1, \ldots, q_{k+1}$ can wait-free simulate a $k$-resilient execution of any asynchronous $n$-process protocol. Informally, the simulation works as follows. Every process $q_i$ tries to simulate steps of all $n$ processes $p_1, \ldots, p_n$ in a round-robin fashion. Simulators run an agreement protocol to make sure that every step is simulated at most once. Simulating a step of a given process may block forever if and only if some simulator has crashed in the middle of the corresponding agreement protocol. Thus, even if $k$ out of $k + 1$ simulators crash, at least $n - k$ simulated processes can still make progress. The simulation thus guarantees at least $n - k$ processes in $\{p_1, \ldots, p_n\}$ accept infinitely many simulated steps.

In the computational component of the reduction algorithm, the BG-simulation technique is used as follows. Let $BG(\mathcal{A}')$ denote the simulation protocol for 2 processes $q_1$ and $q_2$ which allows them to simulate, in a wait-free manner, a 1-resilient execution of algorithm $\mathcal{A}'$ for $n$ processes $p_1, \ldots, p_n$. The complete reduction algorithm thus employs a triple simulation (Figure 3): every process $p_i$ simulates multiple runs of two processes $q_1$ and $q_2$ that use BG-simulation to produce a 1-resilient run of $\mathcal{A}'$ on processes $p_1', \ldots, p_n'$ in which steps of the original algorithm $\mathcal{A}$ are periodically simulated using (ever-growing) DAGs $G_1, \ldots, G_n$. (To avoid confusion, we use $p_j'$ to denote the process that models $p_j$ in a run of $\mathcal{A}'$ simulated by a “real” process $p_i$.)

We are going to use the following property which is trivially satified by BG-simulation:

(BG0) A run of BG-simulation in which every simulator take infinitely many steps simulates a run in which every simulated process takes infinitely many steps.

3.5 Using consensus

The triple simulation we are going to employ faces one complication though. The simulated runs of the asynchronous algorithm $\mathcal{A}'$ may vary depending on which process the simulation is running. This is because $G_1, \ldots, G_n$ are maintained by a parallel computation component (Figure 1), and a process simulating a step of $\mathcal{A}'$ may perform a different number of cycles reading the current version of its DAG before a vertex with desired properties is located (line 13 in Figure 2). Thus, the same sequence of steps of $q_1$ and $q_2$ simulated at different processes may result in different 1-resilient runs of $\mathcal{A}'$: waiting until a vertex $[p_i, d, \ell]$ appears
simulates runs of $BG(A')$ on

simulate runs of $A'$ on

$\begin{array}{c}
\text{Figure 3.} \text{ Three levels of simulation: real processes } p_i \text{ simulate a system of two BG-simulators } q_1 \text{ and } q_2 \text{ that run } BG(A') \text{ to simulate an } (n-1)\text{-resilient run of } A' \text{ on } p'_1, \ldots, p'_n. \\
\end{array}$

\[ r := 0 \]
\[ \text{repeat} \]
\[ \quad r := r + 1 \]
\[ \quad \text{if } G \text{ contains } [p_i, d, \ell] \text{ for some } d \text{ then } u := 1 \]
\[ \quad \text{else } u := 0 \]
\[ \quad v := cons_{i,\ell}.propose(u) \]
\[ \text{until } v = 1 \]

\[ \text{Figure 4.} \text{ Expanded line 13 of Figure 2: waiting until } G \text{ includes a vertex } [p_i, d, \ell] \text{ for some } d. \text{ Here } G \text{ is any DAG generated by the algorithm in Figure 1.} \]

in $G_j$ at process $p_j$ may take different number of local steps checking $G_j$, depending on the time when $p_j$ executes the wait statement in line 13 of Figure 2.

To resolve this issue, the wait statement is implemented using a series of consensus instances $cons_{1,\ell}^i$, $cons_{2,\ell}^i$, $\ldots$ (Figure 4). If $p_i$ is correct, then eventually each correct process will have a vertex $[p_i, d, \ell]$ in its DAG and, thus, the code in Figure 4 is non-blocking, and Theorem 1 still holds. Furthermore, the use of consensus ensures that if a process, while simulating a step of $A'$ at process $p_i$, went through $r$ steps before reaching line 14 in Figure 2, then every process simulating this very step does the same. Thus, a given sequence of steps of $q_1$ and $q_2$ will result in the same simulated 1-resilient run of $A'$, regardless of when and where the simulation is taking place.

### 3.6 Extracting $\Omega$

The computational component of the reduction algorithm is presented in Figure 5. In the component, every process $p_i$ locally simulates multiple runs of a system of 2 processes $q_1$ and $q_2$ that run algorithm $BG(A')$, to produce a 1-resilient run of $A'$ (Figures 2 and 4). Recall that $A'$, in its turn, simulates a run of
Figure 5. Computational component of the reduction algorithm: code for each process $p_i$. Here $ST_A(J, \sigma)$ denotes the state of $A$ reached by the partial run of $A'$ simulated in the partial run of $BG(A')$ with schedule $\sigma$ and input state $J$, and $SCH_{A'}(J, \sigma)$ denotes the corresponding schedule of $A'$.

the original algorithm $A$, using, instead of $D$, the values provided by an ever-growing DAG $G$. In simulating the part of $A'$ of process $p_i'$ presented in Figure 4, $q_1$ and $q_2$ count each access of a consensus instance $\text{cons}_{j,\ell}^{p_i}$ as one local step of $p_i'$ that need to be simulated. Also, in $BG(A')$, when $q_j$ is about to simulate the first step of $p_i'$, $q_j$ uses its own input value as an input value of $p_i'$.

For each simulated state $S$ of $BG(A')$, $p_i$ periodically checks whether the state of $A$ in $S$ is deciding, i.e., whether some process has decided in the state of $A$ in $S$. As we show, eventually, the same infinite non-deciding $1$-resilient run of $A'$ will be simulated by all processes, which allows for extracting the output of $\Omega$.

The algorithm in Figure 5 explores solo extensions of $q_1$ and $q_2$ starting from growing prefixes. Since, by property (BG0) of BG-simulation (Section 3.4), a run of $BG(A')$ in which both $q_1$ and $q_2$ participate infinitely often simulates a run of $A'$ in which every $p_j \in \{p_1', \ldots, p_n'\}$ participates infinitely often, and, by Theorem 1, such a run will produce a fair and thus deciding run of $A$. Thus, if there is an infinite non-deciding run simulated by the algorithm in Figure 2, it must be a run produced by a solo extension of $q_1$ or $q_2$ starting from some finite prefix.

Lemma 2 The algorithm in Figure 5 eventually forever executes lines 23–27.

Proof. Consider any run of the algorithm in Figures 1, 4 and 5. Let $F$ be the failure pattern of that run. Let $G$ be the infinite limit DAG approximated by the algorithm in Figure 1. By contradiction, suppose that lines 23–27 in Figure 5 never block $p_i$.

Suppose that for some initial $J_0$, the call of $explore(J_0, \sigma_0)$ performed by $p_i$ in line 19 never returns. Since the cycle in lines 23–27 in Figure 5 always terminates, there is an infinite sequence of recursive calls $explore(J_0, \sigma_0), explore(J_0, \sigma_1), explore(J_0, \sigma_2), \ldots$, where each $\sigma_\ell$ is a one-step extension of $\sigma_{\ell-1}$. Thus, there exists an infinite never-deciding schedule $\hat{\sigma}$ such that the run of $BG(A')$ based on $\hat{\sigma}$ and $J_0$ produces a never-deciding run of $A'$. Suppose that both $q_1$ and $q_2$ appear in $\hat{\sigma}$ infinitely often. By property (BG0) of BG-simulation (Section 3.4), a run of $BG(A')$ in which both $q_1$ and $q_2$ participate infinitely often simulates
a run of $\mathcal{A}'$ in which every $p_j \in \{p'_1, \ldots, p'_n\}$ participates infinitely often, and, by Theorem 1, such a run will produce a fair and thus deciding run of $\mathcal{A}$ — a contradiction.

Thus, if there is an infinite non-deciding run simulated by the algorithm in Figure 2, it must be a run produced by a solo extension of $q_1$ or $q_2$ starting from some finite prefix. Let $\bar{\sigma}$ be the first such prefix in the order defined by the algorithm in Figure 2 and $q_\ell$ be the first process whose solo extension of $\sigma$ is never deciding. Since the cycle in lines 23–27 always terminates, the recursive exploration of finite prefixes $\sigma$ in lines 28 and 29 eventually reaches $\bar{\sigma}$, the algorithm reaches line 22 with $\sigma = \bar{\sigma}$ and $q_j = q_\ell$. Then the succeeding cycle in lines 23–27 never terminates — a contradiction.

Thus, for all inputs $J_0$, the call of explore($J_0$, $\sigma_0$) performed by $p_i$ in line 19 returns. Hence, for every finite prefix $\sigma$, any solo extension of $\sigma$ produces a finite deciding run of $\mathcal{A}$. We establish a contradiction, by deriving a wait-free algorithm that solves consensus among $q_1$ and $q_2$.

Let $\bar{G}$ be the infinite limit DAG constructed in Figure 1. Let $\beta$ be a map from vertices of $\bar{G}$ to $\mathbb{N}$ defined as follows: for each vertex $[p_i, d, \ell]$ in $G$, $\beta([p_i, d, \ell])$ is the value of variable $r$ at the moment when any run of $\mathcal{A}'$ (produced by the algorithm in Figure 2) exits the cycle in Figure 4, while waiting until $[p_i, d, \ell]$ appears in $G$. If there is no such run, $\beta([p_i, d, \ell])$ is set to 0. Note that the use of consensus implies that if in any simulated run of $\mathcal{A}'$, $[p_i, d, \ell]$ has been found after $r$ iterations, then $\beta([p_i, d, \ell]) = r$, i.e., $\beta$ is well-defined.

Now we consider an asynchronous read-write algorithm $\mathcal{A}'_\beta$ that is defined exactly like $\mathcal{A}'$, but instead of going through the consensus invocations in Figure 4, $\mathcal{A}'_\beta$ performs $\beta([p_i, d, \ell])$ local steps. Now consider the algorithm $BG(\mathcal{A}'_\beta)$ that is defined exactly as $BG(\mathcal{A}')$ except that in $BG(\mathcal{A}'_\beta)$, $q_1$ and $q_2$ BG-simulate runs of $\mathcal{A}'_\beta$. For every sequence $\sigma$ of steps of $q_1$ and $q_2$, the runs of $BG(\mathcal{A}')$ and $BG(\mathcal{A}'_\beta)$ agree on the sequence of steps of $p'_1, \ldots, p'_n$ in the corresponding runs of $\mathcal{A}'$ and $\mathcal{A}'_\beta$, respectively. Moreover, they agree on the runs of $\mathcal{A}$ resulting from these runs of $\mathcal{A}'$ and $\mathcal{A}'_\beta$. This is because the difference between $\mathcal{A}'$ and $\mathcal{A}'_\beta$ consist only in the local steps and does not affect the simulated state of $\mathcal{A}$.

We say that a sequence $\sigma$ of steps of $q_1$ and $q_2$ is deciding with $J_0$, if, when started with $J_0$, the run of $BG(\mathcal{A}'_\beta)$ produces a deciding run of $\mathcal{A}$. By our hypothesis, every eventually solo schedule $\sigma$ is deciding for each input $J_0$. As we showed above, every schedule in which both $q_1$ and $q_2$ appear sufficiently often is deciding by property (BG0) of BG-simulation. Thus, every schedule of $BG(\mathcal{A}'_\beta)$ is deciding for all inputs.

Consider the trees of all deciding schedules of $BG(\mathcal{A}'_\beta)$ for all possible inputs $J_0$. All these trees have finite branching (each vertex has at most 2 descendants) and finite paths. By König’s lemma, the trees are finite. Thus, the set of vertices of $\bar{G}$ used by the runs of $\mathcal{A}'$ simulated by deciding schedules of $BG(\mathcal{A}'_\beta)$ is also finite. Let $G$ be a finite subgraph of $\bar{G}$ that includes all vertices of $\bar{G}$ used by these runs.

Finally, we obtain a wait-free consensus algorithm for $q_1$ and $q_2$ that works as follows. Each $q_j$ runs $BG(\mathcal{A}'_\beta)$ (using a finite graph $G$) until a decision is obtained in the simulated run of $\mathcal{A}$. At this point, $q_j$ returns the decided value. But $BG(\mathcal{A}'_\beta)$ produces only deciding runs of $\mathcal{A}$, and each deciding run of $\mathcal{A}$ solves consensus for inputs provided by $q_1$ and $q_2$ — a contradiction.

\[ \square \]

**Theorem 3** In all environments $\mathcal{E}$, if a failure detector $\mathcal{D}$ can be used to solve consensus in $\mathcal{E}$, then $\Omega$ is weaker than $\mathcal{D}$ in $\mathcal{E}$.

**Proof.** Consider any run of the algorithm in Figures 1, 4 and 5 with failure pattern $F$.

By Lemma 2, at some point, every correct process $p_i$ gets stuck in lines 23–27 simulating longer and longer $q_j$-solo extension of some finite schedule $\sigma$ with input $J_0$. Since, processes $p_1, \ldots, p_n$ use a series of consensus instances to simulate runs of $\mathcal{A}'$ in exactly the same way, the correct processes eventually agree on $\sigma$ and $q_j$. 11
Let \( e \) be the sequence of process identifiers in the 1-resilient execution of \( A' \) simulated by \( q_1 \) and \( q_2 \) in schedule \( \sigma \cdot (q_j) \) with input \( J_0 \). Since a 2-process BG-simulation produces a 1-resilient run of \( A' \), at least \( n - 1 \) simulated processes in \( p_1', \ldots, p_n' \) appear in \( e \) infinitely often. Let \( U (|U| \geq n - 1) \) be the set of such processes.

Now we show that exactly one correct (in \( F \)) process appears in \( e \) only finitely often. Suppose not, i.e., \( \text{correct}(F) \subseteq U \). By Theorem 1, the run of \( A' \) simulated a far run of \( A \), and, thus, the run must be deciding — a contradiction. Since \(|U| \geq n - 1\), exactly one process appears in the run of \( A' \) only finitely often. Moreover, the process is correct.

Thus, eventually, the correct processes in \( F \) stabilize at simulating longer and longer prefixes of the same infinite non-deciding 1-resilient run of \( A' \). Eventually, the same correct process will be observed to take the least number of steps in the run and output in line 26 — the output of \( \Omega \) is extracted. \( \square \)

4 Related Work

Chandra et al. derived the first “weakest failure detector” result by showing that \( \Omega \) is necessary to solve consensus in the message-passing model in their fundamental paper [4]. The result was later generalized to the read-write shared memory model [12, 10].

The technique presented in this paper builds atop two fundamental results. The first is the celebrated BG-simulation [1, 3] that allows \( k + 1 \) processes simulate, in a wait-free manner, a \( k \)-resilient run of any \( n \)-process asynchronous algorithm. The second is a brilliant observation made by Zieliński [15] that any run of an algorithm \( A \) using a failure detector \( D \) induces an asynchronous algorithm that simulates (possibly unfair) runs of \( A \). The recursive structure of the algorithm in Figure 5 is also borrowed from [15]. Unlike [15], however, the reduction algorithm of this paper assumes the conventional read-write memory model without using immediate snapshots [2]. Also, instead of growing “precedence” and ”detector” maps of [15], this paper uses directed acyclic graphs à la [4].

5 Concluding Remarks

This paper presents a new proof that \( \Omega \) is the weakest failure detector to solve consensus in read-write shared memory models. The proof applies a novel reduction technique, and is based on the very fact that wait-free 2-process consensus is impossible, unlike the original technique of [4] that partially rehashes elements of the consensus impossibility proof.

Several recent papers focused on determining the weakest failure detector for a generalization of consensus, \((n,k)\)-set agreement, in which \( n \) processes have to decide on at most \( k \) distinct proposed values [14, 6, 9]. For \( k = 1 \), the problem is consensus, and the weakest failure detector for \((n,1)\)-set agreement is \( \Omega \). Zieliński [15] determined the weakest failure detector for \((n,n-1)\)-set agreement (sometimes called simply set agreement in the literature). However, the case \( 1 < k < n \) remained unresolved until recently.

As we show in the forthcoming paper [8], the technique proposed in this paper can be applied to determine the weakest failure detector for \((n,k)\)-set agreement for any \( 1 \leq k \leq n \). Intuitively, BG simulation allows \( k + 1 \) processes to simulate a \( k \)-resilient run of any asynchronous algorithm, and, generalizing the technique described in this paper, we can derive an infinite non-deciding \( k \)-resilient run \( R \) of \( A' \). At least one correct process appears only finitely often in \( R \) (otherwise, the run would be deciding). Thus, a failure detector that

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1 The result for the shared memory as stated in [12], but the only published proof of it appears in [10].
periodically outputs the latest \( n - k \) processes in growing prefixes of \( R \) guarantees that eventually some correct process is never output. It can be easily shown that this information about failures is sufficient to solve \((n,k)\)-set agreement. For \( k > 1 \), we cannot use consensus to make sure that correct processes simulate runs of \( \mathcal{A}' \) in exactly the same way, regardless of how their local DAGs evolve. Therefore, our generalized reduction algorithm employs a slightly more sophisticated “eventual agreement” mechanism to make sure that the simulation converges.

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**References**


