

(FPT-)Approximation Algorithms for the Virtual Network Embedding Problem

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Abstract

Many resource allocation problems in the cloud can be described as a basic *Virtual Network Embedding Problem* (VNEP): finding mappings of *request graphs* (describing the workloads) onto a *substrate graph* (describing the physical infrastructure). In the offline setting, the two natural objectives are *profit maximization*, i.e., embedding a maximal number of request graphs subject to resource constraints, and *cost minimization*, i.e., embedding all requests at minimal overall cost. Hence, the VNEP can be seen as a generalization of classic routing and call admission problems, in which requests are arbitrary graphs whose communication endpoints are not fixed. Due to its applications, the problem has been studied intensively in the networking community. However, the underlying algorithmic problem is hardly understood.

This paper presents the first fixed-parameter tractable approximation algorithms for the VNEP. Our algorithms are based on randomized rounding. Due to the flexible mapping options and the arbitrary request graph topologies, we show that a novel linear program formulation is required. Only using this novel formulation the computation of convex combinations of valid mappings is enabled, as the formulation needs to account for the structure of the request graphs. Accordingly, to capture the structure of request graphs, we introduce the graph-theoretic notion of extraction orders and extraction width and show that our algorithms have exponential runtime in the request graphs' maximal width. Hence, for request graphs of fixed extraction width, we obtain the first polynomial-time approximations.

Studying the new notion of extraction orders we show that (i) computing extraction orders of minimal width is \mathcal{NP} -hard and (ii) that computing decomposable LP solutions is in general \mathcal{NP} -hard, even when restricting request graphs to planar ones.

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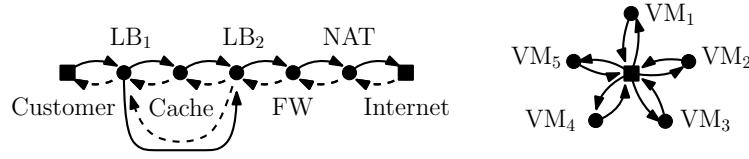
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1 Introduction

At the heart of the cloud computing paradigm lies the idea of efficient resource sharing: due to virtualization, multiple workloads can co-habit and use a given resource infrastructure simultaneously. Indeed, cloud computing introduces great flexibilities in terms of *where* workloads can be mapped and accordingly where resources are allocated. At the same time, exploiting this mapping flexibility poses a fundamental algorithmic challenge.

The underlying algorithmic problem is essentially a graph theoretical one: both the workload as well as the infrastructure can be modeled as *graphs*. The former, the so-called *request graph*, describes the resource requirements both on the nodes (e.g., the virtual machines) as well as on the interconnecting network. The latter, the so-called *substrate graph*, describes the physical infrastructure and its resources (servers and links).

The problem is known in the networking community under the name *Virtual Network Embedding Problem* (VNEP) and has been studied intensively in recent years [8, 13]. The



■ **Figure 1** Examples for virtual networks (i.e., request graphs). Left: A service chain envisioned in 5G networks [19]. Right: a virtual cluster abstraction envisioned in batch processing applications [3].

problem arises in many settings, and is also studied in the realm of embedding service chains [16, 22] and virtual clusters [25].

The online variant in which a minimal cost embedding for a single request is sought after is most prominently studied in the literature. In this work, we study the offline generalization in which multiple requests are given and the objective is to either maximize the profit by selecting a maximal subset of requests to embed or to minimize the cumulative embedding costs. Thus, the offline cost minimization variant reduces to the online problem when considering only a single request.

The design of approximation algorithms for the Virtual Network Embedding Problem has been an open problem for over a decade [8].

1.1 Incarnations of the VNEP in Practice

To highlight the practical relevance of the VNEP, we present two examples in Figure 1. On the left, a *service chain* is depicted, which composes existing network functions (such as a cache, a proxy, or a firewall) into a more advanced network service. The virtualization of network functions enables the faster and more flexible allocation in provider networks [30]. Concretely, the depicted example is envisioned in the context of mobile operators [19]: load-balancers (LB_1, LB_2) are used to route (parts) of the traffic through a cache to optimize the user experience, the firewall (FW) is used to provide security and the network-address translation (NAT) function is used to provide private IP addresses to the customers.

Depicted on the right of Figure 1 is a *virtual cluster*, which was proposed as an abstraction for batch processing applications in the cloud [3]. Concretely, a virtual cluster consists of a set of virtual machines (VMs) and a single logical switch which connects all virtual machines. As originally proposed, all virtual machines and all links have the same computational and bandwidth demands, respectively. The abstraction is attractive due to its simplicity: users only need to specify three numbers, namely the number of VMs together with their uniform demands and the bandwidth to the logical switch.

1.2 Problem Statement

Given is a physical network $G_S = (V_S, E_S)$ offering a set \mathcal{T} of computational types. We refer to the physical network as the *substrate* network. For a type $\tau \in \mathcal{T}$, the set $V_S^\tau \subseteq V_S$ denotes the substrate nodes that can host functionality of the type τ . Denoting the node resources by $R_S^V = \{(\tau, u) \mid \tau \in \mathcal{T}, u \in V_S^\tau\}$ and all substrate resources by $R_S = R_S^V \cup E_S$, the capacity of nodes and edges is denoted by $d_S(x, y) > 0$ for $(x, y) \in R_S$.

The set of request is denoted by \mathcal{R} . For each request $r \in \mathcal{R}$, a directed graph $G_r = (V_r, E_r)$ is given. We refer to the nodes of these graphs as virtual or request nodes and to the edges as virtual or request edges. The type of a virtual node is given via the function $\tau_r : V_r \rightarrow \mathcal{T}$. We allow for node and edge mapping restrictions. Concretely, we assume that the mapping of a virtual node $i \in V_r$ is restricted to a set $V_S^{r,i} \subseteq V_S^{\tau_r(i)}$, while the

mapping of a virtual edge (i, j) is restricted to a set $E_S^{r,i,j} \subseteq E_S$. Each virtual node $i \in V_r$ and each edge $(i, j) \in E_r$ is attributed with a resource demand $d_r(i) \geq 0$ and $d_r(i, j) \geq 0$, respectively. Virtual nodes and edges can only be mapped on substrate nodes and edges of sufficient capacity and we have $V_S^{r,i} \subseteq \{u \in V_S^{r,i} | d_S(u) \geq d_r(i)\}$ and $E_S^{r,i,j} \subseteq \{(u, v) \in E_S | d_S(u, v) \geq d_r(i, j)\}$. We denote by $d_{\max}(r, x, y)$ the maximal demand that a request r may impose on a resource $(x, y) \in R_S$, i.e. $d_{\max}(r, \tau, u) = \max_{i \in V_r, u \in V_S^{r,i}} d_r(i)$ for $(\tau, u) \in R_S^V$ and $d_{\max}(r, u, v) = \max_{(i,j) \in E_r, (u,v) \in E_S^{r,i,j}} d_r(i, j)$ for $(u, v) \in E_S$.

The mapping of a request onto the substrate graph is captured by the following definition.

► **Definition 1 (Valid Mapping).** A valid mapping m_r of request $r \in \mathcal{R}$ is a tuple (m_r^V, m_r^E) of functions $m_r^V : V_r \rightarrow V_S$ and $m_r^E : E_r \rightarrow \mathcal{P}(E_S)$, such that the following conditions hold:

- Virtual nodes are mapped to allowed substrate nodes: $m_r^V(i) \in V_S^{r,i}$ holds for all $i \in V_r$.
- The mapping $m_r^E(i, j)$ of virtual edge $(i, j) \in E_r$ is a path connecting $m_r^V(i)$ to $m_r^V(j)$, which only uses allowed edges, i.e., $m_r^E(i, j) \subseteq \mathcal{P}(E_S^{r,i,j})$ holds.

We denote by \mathcal{M}_r the set of *all* valid mappings of request $r \in \mathcal{R}$. □

Note that the edge mapping $m_r^E(i, j)$ may be empty, iff. $m_r^V(i) = m_r^V(j)$ holds for edges $(i, j) \in E_r$. Next, we introduce the notion of allocations induced by a valid mapping.

► **Definition 2 (Allocations of a Valid Mapping).** The allocation $A(m_r, x, y)$ induced by mapping m_r on resource $(x, y) \in R_S$ is defined as follows: $A(m_r, \tau, u) = \sum_{i \in V_r, \tau(i)=\tau, m_r^V(i)=u} d_r(i)$ holds for $(\tau, u) \in R_S^V$ and $A(m_r, u, v) = \sum_{(i,j) \in E_r, (u,v) \in m_r^E(i,j)} d_r(i, j)$ holds for $(u, v) \in E_S$, respectively. The maximal allocation that a valid mapping of request r may impose on a substrate resource $(x, y) \in R_S$ is denoted by $A_{\max}(r, x, y) = \max_{m_r^k \in \mathcal{M}_r} A(m_r, x, y)$. □

Given a collection of valid mappings $\{m_r\}_{r \in \mathcal{R}'}$ for a subset of requests $\mathcal{R}' \subseteq \mathcal{R}$, we refer to this collection as *feasible* if the cumulative allocations obey substrate capacities, i.e., $\sum_{r \in \mathcal{R}'} A(m_r, x, y) \leq d_S(x, y)$ holds for all resources $(x, y) \in R_S$.

► **Definition 3 (Virtual Network Embedding Problem).** The profit variant of the Virtual Network Embedding Problem (VNEP) asks for finding a feasible collection $\{m_r\}_{r \in \mathcal{R}'}$ of mappings while maximizing the overall profit $\sum_{r \in \mathcal{R}'} b_r$, where $b_r > 0$ denotes the benefit obtained for embedding request r . For the cost variant all of the given requests \mathcal{R} must be feasibly embedded while minimizing the resource costs $\sum_{(x,y) \in R_S} c_S(x, y) \cdot \sum_{r \in \mathcal{R}} A(m_r, x, y)$, where $c_S(x, y) \geq 0$ denotes the resource cost of $(x, y) \in R_S$. □

Formulation 1: Enumerative Formulation for the VNEP (left: profit, right: cost)

$$(1) \quad \max \sum_{r \in \mathcal{R}, m_r \in \mathcal{M}_r} f_r^k \cdot b_r \quad \min \sum_{(x,y) \in R_S} c_S(x, y) \sum_{r \in \mathcal{R}, m_r \in \mathcal{M}_r} f_r^k \cdot A(m_r, x, y) \quad (2)$$

$$(3) \quad \sum_{m_r^k \in \mathcal{M}_r} f_r^k \leq 1 \quad \forall r \in \mathcal{R} \quad \sum_{m_r^k \in \mathcal{M}_r} f_r^k = 1 \quad \forall r \in \mathcal{R} \quad (4)$$

$$\sum_{r \in \mathcal{R}, m_r^k \in \mathcal{M}_r} f_r^k \cdot A(m_r^k, x, y) \leq d_S(x, y) \quad \forall (x, y) \in R_S \quad (5)$$

The VNEP can be expressed as the non-polynomial sized Formulation 1, which explicitly enumerates *all* valid mappings: for each request $r \in \mathcal{R}$ and each mapping $m_r^k \in \mathcal{M}_r$ a variable f_r^k is introduced. Setting $f_r^k \in \{0, 1\}$ yields integer programs and setting $f_r^k \in [0, 1]$

yields the respective linear programs. We refer to the problem over the linear variables, in which convex combinations of valid mappings are allowed, as the *fractional* VNEP.

The Virtual Network Embedding Problem is known to be strongly \mathcal{NP} -hard [27].

1.3 Putting the VNEP Into Perspective

The VNEP can be seen as a generalization of many well-studied problems. The profit variant is e.g. related to *routing requests* [2, 6] and *virtual circuits* [1, 20], and the *unsplittable flow problem* [4], while the cost variant is related to the shortest k -disjoint paths problem [7, 24], and the subgraph isomorphism problem [10].

The most notable differences to the aforementioned problems are (i) that request node locations are not fixed a priori and that (ii) a single request represents a graph instead of e.g. a single link as in the unsplittable flow problem. Accordingly, the key challenge we face when designing approximation algorithms, is that virtual nodes can in principle be mapped on *any* substrate node and each virtual edge may traverse any substrate edge.

1.4 Our Results and Techniques

In this paper we set out to initiate the study of approximation algorithms for the VNEP for *arbitrary* request graphs. Leveraging the VNEP’s connection to multi-commodity flow problems, we employ randomized rounding to obtain our results. This technique has proven both simple and effective: given an Integer Program (IP) for a problem, solutions of its Linear Program (LP) are decomposed into convex combinations (cf. Formulation 1) and then *rounded* according to their weight.

While in many contexts the natural LP, obtained by relaxing the corresponding integer program, is sufficiently strong to extract convex combinations of solutions, this is not the case for the VNEP. As the mapping of flow endpoints is flexible in the VNEP, we prove that the natural Multi-Commodity Flow (MCF) formulation for the VNEP fails to ensure the decomposability into convex combinations. In fact, it fails to capture the structure of valid mappings and we prove that the MCF formulation’s integrality gap is unbounded.

Analyzing the shortcomings of the MCF formulation, we obtain sufficient conditions to ensure decomposability. Accordingly, we develop a novel LP formulation for the VNEP which incorporates the requests’ individual structure. The dependency of our formulation on the underlying request graphs comes at the price that the size of the formulation grows exponentially in the ‘complexity’ of the request graphs. Our formulation relies on acyclic (re-)orientations of request graphs called *extraction orders* to guide the process of extracting valid mappings. Based on confluences in these extraction orders, i.e. disjoint paths, we introduce the notion of *extraction width*. In turn, we show that the size of our LP formulation, and hence also the runtime of our approximations, are fixed-parameter tractable (FPT) in the *extraction width* of the given *extraction orders*.

Hence, finding efficient approximations boils down to finding extraction orders of small width. Our initial results are quite intriguing: we show that depending on the chosen extraction order the width can differ by a factor of $\Omega(|G_r|)$ (which is maximal) and that finding the minimal extraction width is itself \mathcal{NP} -hard. While this may raise questions about the sensibility of our graph-theoretic notions, we also show that there cannot exist any polynomial-time algorithm (neither linear nor combinatorial) that can solve the fractional VNEP (even when restricting the requests to planar graphs).

Having set out to obtain approximations for the VNEP, we eventually derive the first (FPT-)approximations for the profit and cost variants of the VNEP for arbitrary request

graphs by using our novel LP formulation. The presented approximations provide constant approximation guarantees for the cost and the profit while exceeding resource capacities by a factor of $\mathcal{O}(1 + \varepsilon \cdot \sqrt{2 \cdot \Delta(R_S) \cdot \log |V_S|})$, where $\varepsilon \leq 1$ is the ratio of maximal demand to minimal capacity and $\Delta(R_S) = \max_{(x,y) \in R_S} \sum_{r \in \mathcal{R}} \left(\frac{A_{\max}(r,x,y)}{d_{\max}(r,x,y)} \right)^2$ captures the (sum of squared) ratios of the maximal cumulative allocation divided by the maximal allocation.

1.5 Related Work

In the last decade, the VNEP has attracted much attention due to its numerous applications. A survey from 2013 lists more than 80 different algorithms for its many variations [13]. A large fraction of the existing literature considers heuristics without giving approximation guarantees [8, 31]. Other works proposed exact methods as integer or constraint programming, coming at the cost of an exponential runtime [21, 29].

In contrast, we initiate the study of (FPT-)approximation algorithms for the VNEP with provable approximation guarantees for *arbitrary substrate and request graphs*. The works closest to ours are by Even et al. [11, 12] and Bansal et al. [5]. Even et al. studied approximation algorithms and competitive online algorithms for the embedding of request chains. Bansal et al. consider the setting of embedding tree request graphs under the objective to minimize the maximum load and also provide approximations and competitive online algorithms. Their main result is a $n^{O(d)}$ time $O(d^2 \log(nd))$ -approximation algorithm for the embedding of a single tree of depth d on a substrate with n nodes, which is based on a strong LP relaxation inspired by the Sherali-Adams hierarchy. By considering only tree requests, Bansal et al. do not address the problem of computing valid mappings for request graphs containing cycles. However, and importantly, the approach of Bansal et al. is complementary to ours and may hence potentially be combined with our results in the future to obtain stronger approximations and also derive competitive online algorithms.

Bibliographic Note. This work significantly extends the authors' previous technical report [26] as well as the publication [28], which only consider approximation algorithms for cactus request graphs and are hence not applicable for arbitrary request graphs.

1.6 Organization

The remainder of this paper is organized as follows. Section 2 studies the classic multi-commodity flow formulation and shows its limitations. In Section 3 we present our decomposable LP formulation and introduce graph-theoretic notions as extraction confluences and extraction width. In Section 4 we present our FPT-approximations for the VNEP. In Section 5 we shortly study properties of the novel extraction width concept and show that cactus request graphs have a constant extraction width. We conclude our paper in Section 6.

2 Limitations of Classic Multi-Commodity Formulations for VNEP

In this section, we study the Multi-Commodity Flow (MCF) formulation for solving the VNEP (see Formulation 2), which is widely used [8, 17, 27, 29]. We first show the positive result that the formulation is sufficiently strong to compute solutions to the fractional VNEP when requests are *trees*. Subsequently, we show that the formulation fails to allow for the decomposition of *cyclic* request graphs into convex combinations of valid mappings.

Formulation 2: Multi-Commodity Flow Base Formulation for the VNEP

$$\begin{aligned} \sum_{u \in V_S^{r,i}} y_{r,i}^u &= x_r & \forall r \in \mathcal{R}, i \in V_r & \quad (6) \\ y_{r,i}^u &= 0 & \forall r \in \mathcal{R}, i \in V_r, u \in V_S \setminus V_S^{r,i} & \quad (7) \\ \sum_{(u,v) \in \delta^+(u)} z_{r,i,j}^{u,v} - \sum_{(v,u) \in \delta^-(u)} z_{r,i,j}^{v,u} &= y_{r,i}^u - y_{r,j}^u & \forall r \in \mathcal{R}, (i,j) \in E_r, u \in V_S & \quad (8) \\ z_{r,i,j}^{u,v} &= 0 & \forall r \in \mathcal{R}, (i,j) \in E_r, (u,v) \in E_S \setminus E_S^{r,i,j} & \quad (9) \\ \sum_{(i,j) \in E_r} d_r(i,j) \cdot z_{r,i,j}^{u,v} &= a_r^{u,v} & \forall r \in \mathcal{R}, (u,v) \in E_S & \quad (10) \\ \sum_{i \in V_r, \tau_r(i)=\tau} d_r(i) \cdot y_{r,i}^u &= a_r^{\tau,u} & \forall r \in \mathcal{R}, (\tau, u) \in R_S^V & \quad (11) \\ \sum_{r \in \mathcal{R}} a_r^{x,y} &\leq d_S(x,y) & \forall (x,y) \in R_S & \quad (12) \end{aligned}$$

2.1 The Multi-Commodity Formulation

We explain the formulation by considering its integer variant. The variable $x_r \in \{0,1\}$ indicates whether request $r \in \mathcal{R}$ is embedded or not. The variable $y_{r,i}^u \in \{0,1\}$ indicates whether virtual node $i \in V_r$ is mapped on substrate node u . Similarly, the flow variable $z_{r,i,j}^{u,v} \in \{0,1\}$ indicates whether the substrate edge $(u,v) \in E_S$ is part of the path of the virtual edge $(i,j) \in E_r$. The variable $a_r^{x,y} \geq 0$ denotes the cumulative allocations that the embedding of request r induces on resource $(x,y) \in R_S$.

By Constraints 6 and Constraint 7, virtual nodes are only mapped on suitable substrate nodes when $x_r = 1$ holds. Constraint 8 induces an unsplitable unit flow for each virtual edge $(i,j) \in E_r$ from the substrate location onto which i was mapped to the substrate location onto which j was mapped. By Constraint 9 the mapping of virtual edges may only consist of *allowed* substrate edges. Constraints 10 and 11 compute the cumulative allocations while Constraint 12 enforces that resource capacities are respected. Applying the objective $\max \sum_{r \in \mathcal{R}} b_r \cdot x_r$ the profit variant is obtained. Setting $\min \sum_{(x,y) \in R_S, r \in \mathcal{R}} c_S(x,y) \cdot a_r^{x,y}$ and enforcing $x_r = 1$ for all requests $r \in \mathcal{R}$ the cost variant is obtained.

The LP formulation is obtained by relaxing the domain of the above introduced binary variables to $[0,1]$. The following lemma states that whenever a virtual node $i \in V_r$ is (fractionally) mapped on a certain substrate node, suitable mappings for all incident edges and their endpoints can be found.

► Lemma 4 (Local Connectivity Property of the MCF Formulation).

Consider a fractional solution $(x_r, \vec{y}_r, \vec{z}_r, \vec{a}_r)$ to the LP Formulation 2 for request $r \in \mathcal{R}$. If $y_{r,i}^u > 0$ holds for $i \in V_r$ and $u \in V_S^{r,i}$, then for incoming edges $(k,i) \in E_r$ and outgoing edges $(i,j) \in E_r$ there exist substrate paths $P_{r,k,i}^{v,u}$ and $P_{r,i,j}^{u,w}$, such that:

1. $P_{r,k,i}^{v,u}$ is a path from v to u , such that $y_{r,k}^v > 0$ and $z_{r,k,i}^e > 0$ holds for $e \in P_{r,k,i}^{v,u}$.
2. $P_{r,i,j}^{u,w}$ is a path from u to w , such that $y_{r,j}^w > 0$ and $z_{r,i,j}^e > 0$ holds for $e \in P_{r,i,j}^{u,w}$.

The respective paths $P_{r,k,i}^{v,u}$ and $P_{r,i,j}^{u,w}$ can be found in time $\mathcal{O}(|E_S|)$ by a simple graph search.

Proof. Fix any substrate node $u \in V_S$ for which $y_{r,i}^u > 0$ holds. We first consider the outgoing edges $(i,j) \in E_r$. By Constraint 6, $\sum_{u \in V_S^{r,i}} y_{r,i}^u = \sum_{v \in V_S^{r,j}} y_{r,j}^v$ holds. Hence, the virtual node $j \in V_r$ must be mapped also at least with value $y_{r,i}^u$. If j is also partially mapped on u , i.e., if $y_{r,j}^u > 0$ holds, then the result follows directly, as u connects to u using (and

allowing) the empty path $P_{r,i,j}^{u,u} = \langle \rangle$. If, on the other hand, $y_{r,j}^u = 0$ holds, then Constraint 8 induces an flow of value $y_{r,i}^u$ at substrate node u with respect to the commodity $z_{r,i,j}$. As the right hand side of Constraint 8 may only attain negative values at nodes $w \in V_S^{r,j}$ for which $y_{r,j}^w > 0$ holds, the flow (of commodity $z_{r,i,j}$) emitted at node u must eventually reach a node $w \in V_S^{r,j}$ with $y_{r,j}^w > 0$ and hence the result follows for any outgoing edge $(i,j) \in E_r$. Note that the corresponding path $P_{r,i,j}^{u,w}$ can be constructed in time $\mathcal{O}(|E_S|)$ by a simple breadth-first search, which only considers edges $(u',v') \in E_S$ for which $z_{r,i,j}^{u',v'} > 0$ holds.

The argument for incoming edges $(k,i) \in E_r$ is the same and the respective paths $P_{r,k,i}^{v,u}$ can be recovered by breadth-first searches traversing substrate edges $(u,v) \in E_S$ in their opposite direction when $z_{r,k,i}^{u,v} > 0$ holds. ■

2.2 Decomposing Solutions to the MCF Formulation

Given the connectivity property of Lemma 4, we argue how solutions to the LP relaxation of the MCF formulation can be decomposed into convex combinations $\mathcal{D}_r = \{(f_r^k, m_r^k)\}_k$ (cf. LP Formulation 1) *as long as the request graphs are trees*. The ideas presented henceforth will also apply for the decomposition of our novel formulation presented in Section 3.

We naturally apply the idea of Ford and Fulkerson [14] for decomposing $s-t$ flows into paths to our setting. Given a LP solution $(x_r, \vec{y}_r, \vec{z}_r, \vec{a}_r)$ for request $r \in \mathcal{R}$, we need to find a valid mapping $m_r = (m_r^V, m_r^E) \in \mathcal{M}_r$ which is *covered* by the embedding variables. Concretely, letting $\mathcal{V}(m_r) = \{y_{r,i}^{m_r^V(i)} \mid i \in V_r\} \cup \{z_{r,i,j}^{u,v} \mid (i,j) \in E_r, (u,v) \in m_r^E(i,j)\}$ denote all the LP variables involved under mapping m_r , we say that the mapping m_r is covered by the LP solution iff. $f_r = \min \mathcal{V} \Rightarrow 0$ holds. Accordingly, the mapping m_r of weight f_r can be *extracted* by reducing the variables in \mathcal{V} by f_r while adding (f_r, m_r) to the set of convex combinations \mathcal{D}_r . Importantly, after the extraction, the now adapted LP solution is still feasible and hence the extraction process can be repeated. To find a mapping in the first place, the mapping of nodes and edges has to be done in some order. We refer to this order as the extraction order:

► **Definition 5** (Extraction Order $G_r^{\mathcal{X}}$). Given a virtual network $G_r = (V_r, E_r)$, we refer to any rooted graph $G_r^{\mathcal{X}} = (V_r, E_r^{\mathcal{X}}, s_r)$ as an *extraction order*, if the following holds:

1. $G_r^{\mathcal{X}}$ is a directed acyclic graph, s.t. each node is reachable from the root $s_r \in V_r$, and
2. $E_r^{\mathcal{X}}$ is obtained from E_r by (potentially) reversing the orientation of some edges.

We denote by $\vec{E}_r : E_r^{\mathcal{X}} \rightarrow E_r$ the function yielding the edge's original orientation and by $\overleftarrow{E}_r^{\mathcal{X}} : E_r \rightarrow E_r^{\mathcal{X}}$ its inverse. We write $\delta_{\vec{\mathcal{X}}}^+(i) = \{(i,j) \in E_r^{\mathcal{X}}\}$ and $\delta_{\overleftarrow{\mathcal{X}}}^-(i) = \{(j,i) \in E_r^{\mathcal{X}}\}$ to denote the outgoing and incoming edges with respect to the edge set $E_r^{\mathcal{X}}$. □

Given the extraction order $G_r^{\mathcal{X}}$, the extraction process works by first choosing a suitable mapping location for the root s_r . Given this location, Lemma 4 is applied to obtain mappings for all outgoing edges of s_r (according to $E_r^{\mathcal{X}}$) *together* with their heads. Continuing to apply Lemma 4 for each of the newly mapped nodes, a complete mapping in which all virtual nodes and edges are mapped on suitable substrate nodes and edges is constructed.

Algorithm 1 formalizes the decomposition scheme to extract convex combinations of valid mappings from solutions to Formulation 2. The algorithm extracts mappings m_r^k of value f_r^k iteratively, as long as $x_r > 0$ holds. Initially, in the k -th iteration, none of the virtual nodes and edges are mapped. As $x_r > 0$ holds, the root node s_r must be mapped accordingly by Constraint 6, i.e. there must exist a node $u \in V_S^{r,s_r}$ with $y_{r,s_r}^u > 0$ and the algorithm sets $m_r^V(s_r) = u$. Given this initial fixing, the algorithm iteratively extracts nodes from the queue

Algorithm 1: Decomposition algorithm of MCF solutions for Tree Requests

Input : Tree request $r \in \mathcal{R}$ together with a solution $(x_r, \vec{y}_r, \vec{z}_r, \vec{a}_r)$ for Formulation 2
Extraction order $G_r^{\mathcal{X}} = (V_r, E_r^{\mathcal{X}}, s_r)$

Output : Convex combination $\mathcal{D}_r = \{D_r^k = (f_r^k, m_r^k)\}_k$ of valid mappings

- 1 set $\mathcal{D}_r \leftarrow \emptyset$ and $k \leftarrow 1$
- 2 while $x_r > 0$ do
- 3 set $m_r^k \leftarrow (m_r^V, m_r^E) \leftarrow (\emptyset, \emptyset)$
- 4 set $\mathcal{Q} \leftarrow \{s_r\}$
- 5 choose $u \in V_S^{r, s_r}$ with $y_{r, s_r}^u > 0$ and set $m_r^V(s_r) \leftarrow u$
- 6 while $|\mathcal{Q}| > 0$ do
- 7 choose $i \in \mathcal{Q}$ and set $\mathcal{Q} \leftarrow \mathcal{Q} \setminus \{i\}$
- 8 foreach $(i, j) \in \vec{\delta}_{E_r^{\mathcal{X}}}(i)$ do
- 9 if $(i, j) = \vec{E}_r(i, j)$ then
- 10 compute path $P_{r, i, j}^{u, v}$ from $m_r^V(i) = u$ to $v \in V_S^{r, j}$ according to Lemma 4
such that $y_{r, j}^v > 0$ and $z_{r, i, j}^{u', v'} > 0$ hold for $(u', v') \in P_{r, i, j}^{u, v}$
- 11 set $m_r^V(j) \leftarrow v$ and $m_r^E(i, j) \leftarrow P_{r, i, j}^{u, v}$
- 12 else
- 13 compute path $P_{r, j, i}^{v, u}$ from $v \in V_S^{r, j}$ to $m_r^V(i) = u$ according to Lemma 4
such that $y_{r, j}^v > 0$ and $z_{r, j, i}^{u', v'} > 0$ hold for $(u', v') \in P_{r, j, i}^{v, u}$
- 14 set $m_r^V(j) \leftarrow v$ and $m_r^E(\vec{E}_r(i, j)) \leftarrow P_{r, j, i}^{u, v}$
- 15 set $\mathcal{Q} \leftarrow \mathcal{Q} \cup \{j\}$
- 16 set $\mathcal{V}_k \leftarrow \{x_r\} \cup \{y_{r, i}^{m_r^V(i)} | i \in V_r\} \cup \{z_{r, i, j}^{u, v} | (i, j) \in E_r, (u, v) \in m_r^E(i, j)\}$
- 17 set $f_r^k \leftarrow \min \mathcal{V}_k$
- 18 set $v \leftarrow v - f_r^k$ for all $v \in \mathcal{V}_k$ and set $a_r^{x, y} \leftarrow a_r^{x, y} - f_r^k \cdot A(m_r^k, x, y)$ for all $(x, y) \in R_S$
- 19 add $D_r^k = (f_r^k, m_r^k)$ to \mathcal{D}_r and set $k \leftarrow k + 1$
- 20 return \mathcal{D}_r

\mathcal{Q} which have already been mapped and considers all outgoing virtual edges $(i, j) \in E_r^{\mathcal{X}}$. If the orientation of edge (i, j) was not changed, i.e., if $(i, j) = \vec{E}_r(i, j)$ holds, then Lemma 4 is applied to obtain a mapping of the edge (i, j) together with its head j . If the edge's orientation was reversed, i.e. iff. $(i, j) \neq \vec{E}_r(i, j)$ holds, Lemma 4 can be applied again, only now a path from the mapping of the head i (according to the edge's original orientation) to some mapping of the tail j is obtained. Lastly, the minimum mapping value f_r^k is computed and the variables of the LP (including the allocation variables) are decreased accordingly. The formal correctness of the algorithm is proven in Lemma 6.

► **Lemma 6.** *Given a virtual network request $r \in \mathcal{R}$, whose underlying undirected graph is a tree, and a solution $(x_r, \vec{y}_r, \vec{z}_r, \vec{a}_r)$ to the LP Formulation 2, the solution can be decomposed into convex combinations of valid mappings $\mathcal{D}_r = \{(f_r^k, m_r^k)\}_k$, such that the following holds:*

- The decomposition is complete, i.e., $x_r = \sum_k f_r^k$ holds.
- The decomposition's resource allocations are bounded by \vec{a}_r , i.e., $a_r^{x, y} \geq \sum_k f_r^k \cdot A(m_r^k, x, y)$ holds for each resource $(x, y) \in R_S$.

Proof. Note that the mapping of each virtual node and each virtual edge is valid by construction: Constraints (7) and (9) enforce that a node and an edge can only be mapped in a valid fashion. Furthermore, as $G_r^{\mathcal{X}}$ is an arborescence, node mappings are never revoked and each node of G_r will eventually be mapped. The mapping value f_r^k is computed as the

minimum of the mapping variables \mathcal{V}_k used for constructing m_r^k . Reducing the values of the mapping variables together with the load variables \bar{a}_r , the Constraints 6-10 continue to hold.

As the decomposition process continues as long as $x_r > 0$ holds and in the k -th step at least one variable's value is set to 0, it is easy to check that (i) the algorithm terminates with a complete decomposition for which $\sum_k f_r^k = x_r$ holds and (ii) the algorithm has polynomial runtime, as the number of variables for request r is bounded by $\mathcal{O}(|G_r| \cdot |E_S|)$. ■

2.3 Limitations of the MCF Formulation

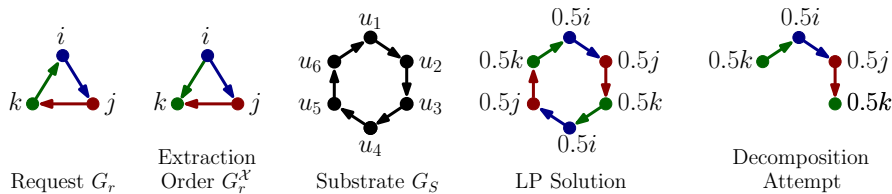
Having shown the decomposability of LP solutions for tree requests, we now show that this does not hold, if the request graphs contain *cycles*. Figure 2 gives an example for an LP solution of Formulation 2 from which no valid mapping (that is covered) can be extracted. Concretely, considering the mapping of i on u_1 and following the depicted extraction order, k and j must be mapped on u_6 and u_2 , respectively. However, the mapping of j on u_2 only allows for the mapping of k on u_3 and no valid mapping can be extracted and we obtain:

► **Theorem 7.** *Solutions to the LP Formulation 2 can (in general) not be decomposed into convex combinations of valid mappings, if request graphs contain cycles. Accordingly, the integrality gap of the LP Formulation 2 is unbounded for cyclic request graphs.*

Proof. Figure 2 depicts an example solution to the LP Formulation 2 from which *not a single* valid mapping can be extracted. The validity of the depicted solution is easy to check. As virtual node $i \in V_r$ is mapped onto substrate node $u_1 \in V_S$, and $u_2 \in V_S$ is the only neighboring node with respect to the commodity $z_{r,i,j}$ that hosts $j \in V_r$, a mapping (m_r^V, m_r^E) with $m_r^V(i) = u_1$ and $m_r^V(j) = u_2$ must exist. Similarly, $m_r^V(k) = u_3$ must hold. However, the flow of virtual edge $(k, i) \in E_r$ leaving $u_3 \in V_S$ only leads to $u_4 \in V_S$. Hence the virtual node $i \in V_r$ must be mapped both on u_1 and u_4 . As the same argument applies when considering the mapping of i onto u_4 , no valid mapping can be extracted.

We now show that the formulation exhibits an unbounded integrality gap. Consider the following restrictions for mapping the virtual links: $E_S^{r,i,j} = \{(u_1, u_2), (u_4, u_5)\}$, $E_S^{r,j,k} = \{(u_2, u_3), (u_5, u_6)\}$, $E_S^{r,k,i} = \{(u_3, u_4), (u_6, u_1)\}$. Note that the solution depicted in Figure 2 is still feasible for the MCF LP. Considering the profit variant of the MCF formulation, the LP will attain an objective of b_r . As on the other hand, there does not exist a valid mapping of request r on G_S , the optimal solution achieves a profit of 0. Hence, the integrality gap of the profit formulation is unbounded.

For the cost variant, we add an edge (u_3, u_1) of arbitrarily high cost to the substrate and include this edge in the set of allowed edges for the virtual edge $(k, i) \in E_r$. Hence, there exists only a single valid mapping, which uses this edge (u_3, u_1) while the MCF formulation



■ **Figure 2** Example showing that solutions to the LP Formulation 2 cannot be decomposed into convex combinations of valid mappings. The LP solution with $x_r = 1$ is depicted as follows. Substrate nodes are annotated with virtual node mappings: $0.5i$ at node u_1 indicates $y_{r,i}^{u_1} = 1/2$. Substrate edge colors match the color of the virtual edges mapped to it. All virtual edges are also mapped using flow values $1/2$. The color of substrate edge (u_1, u_2) therefore implies that $z_{r,i,j}^{u_1, u_2} = 1/2$ holds.

might still use the LP solution depicted in Figure 2. Hence, as the cost of the edge (u_3, u_1) can be arbitrarily high, the integrality gap is unbounded. \blacksquare

3 Novel Decomposable LP Formulation

In this section we present our novel LP formulation to solve the fractional VNEP and is the basis for our randomized rounding approximation algorithms for the VNEP. We first present the high-level idea of our formulation and introduce crucial concepts as extraction confluences and the extraction width. After introducing further notation, we formally present the LP formulation and show the decomposability of its solutions.

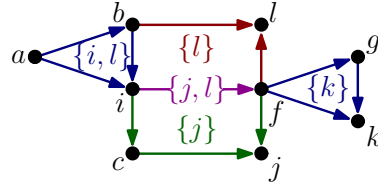
3.1 Idea and Definitions

We shortly outline the key idea of our formulation by analyzing the shortcomings of the MCF formulation. Considering the example of Figure 2, we observe that there exist two virtual paths towards k in $G_r^{\mathcal{X}}$, namely $\langle(i, k)\rangle$ and $\langle(i, j), (j, k)\rangle$. We refer to the combination of two such paths in $G_r^{\mathcal{X}}$ leading from a common virtual node to another common node as an *extraction confluence*:

► **Definition 8** (Extraction Confluence $C_{i,j}^{\mathcal{X}}$). Given an extraction order $G_r^{\mathcal{X}}$, an *extraction confluence* $C_{i,j}^{\mathcal{X}} = P_{i,j}^1 \sqcup P_{i,j}^2$ connects $i \in V_r$ to $j \in V_r$ using two otherwise node-disjoint paths $P_{i,j}^1, P_{i,j}^2 \subseteq E_r^{\mathcal{X}}$. We refer to i as the source and to j as the target of the confluence $C_{i,j}^{\mathcal{X}}$. \square

According to the connectivity property of the MCF formulation (cf. Lemma 4), (partial) mappings can always be extended, but the disjoint paths of a confluence might lead to *different* mappings of the confluence's target as depicted in Figure 2. However, this *divergence* is only possible when the confluence's target can be mapped on multiple locations and is not fixed.

We use this as follows. Considering a confluence $C_{i,j}^{\mathcal{X}}$, our LP formulation considers multiple copies of the MCF formulation *for each potential mapping location of the confluence's target*. In each of these copies, the mapping of the confluence's target is fixed to a *specific* substrate node. To generalize this idea to multiple confluences, we label edges with confluence targets as follows.



► **Figure 3** Exemplary labeled $G_r^{\mathcal{X}}$.

► **Definition 9** (Extraction Edge Labels). We introduce edge labels $\mathcal{L}_{r,e}^{\mathcal{X}} \subseteq V_r$ for $e \in E_r^{\mathcal{X}}$ as follows. The extraction order edge e is labeled with node j , i.e., $j \in \mathcal{L}_{r,e}^{\mathcal{X}}$ holds, iff. a confluence $C_{i,j}^{\mathcal{X}}$ with target j exists that contains e . We also label the edges in their original orientation accordingly: for edge $e \in E_r$ we set $\mathcal{L}_{r,e} \triangleq \mathcal{L}_{r,e'}^{\mathcal{X}}$ with $e' = \overrightarrow{E}_r^{\mathcal{X}}(e)$. \square

The edge labels will be used in our novel LP formulation to instantiate copies of the MCF formulation. Additionally, we introduce *confluence edge bags* which partition outgoing edges.

► **Definition 10** (Confluence Edge Bags). Given an extraction order $G_r^{\mathcal{X}}$, the outgoing edges $\delta_{\mathcal{X}}^+(i)$ of each node $i \in V_r$ are partitioned into a set of edge bags $\mathcal{B}_{r,i}^{\mathcal{X}} = \{B_{r,i}^{\mathcal{X},b}\}_b$, such that two edges $e_1, e_n \in \delta_{\mathcal{X}}^+(i)$ are placed in the same bag $B_{r,i}^{\mathcal{X},b}$, iff. there exists a series of edges $e_2, e_3 \dots, e_{n-1} \in \delta_{\mathcal{X}}^+(i)$ such that $\mathcal{L}_{r,e_l}^{\mathcal{X}} \cap \mathcal{L}_{r,e_{l+1}}^{\mathcal{X}} \neq \emptyset$ holds for $l \in \{1, \dots, n-1\}$.

We denote by $\mathcal{L}_{r,i}^{\mathcal{X},b} = \bigcup_{e \in B_{r,i}^{\mathcal{X},b}} \mathcal{L}_{r,e}^{\mathcal{X}}$ the union of labels contained in a bag $B_{r,i}^{\mathcal{X},b} \in \mathcal{B}_{r,i}^{\mathcal{X}}$ and by $\mathcal{L}_{b \cap e}^{\mathcal{X}} = \mathcal{L}_{r,e}^{\mathcal{X}} \cap \mathcal{L}_{r,i}^{\mathcal{X},b}$ the intersection of labels of the bag $B_{r,i}^{\mathcal{X},b}$ and the edge $e \in E_r^{\mathcal{X}}$. \square

The size of our formulation will be proven to be exponential in the maximal number of labels contained in any edge bag, and we define the notion of *extraction width* accordingly:

► **Definition 11** (Extraction Width). The width $\text{ew}_{\mathcal{X}}(G_r^{\mathcal{X}})$ of a given extraction order $G_r^{\mathcal{X}}$ is the maximal number of labels contained in an edge bag plus one: $\text{ew}_{\mathcal{X}}(G_r^{\mathcal{X}}) = 1 + \max_{i \in V_r, B_{r,i}^{\mathcal{X}}, b \in \mathcal{B}_{r,i}^{\mathcal{X}}} |\mathcal{L}_{r,i}^{\mathcal{X},b}|$. Denoting by $\mathcal{X}(G_r)$ the set of all extraction orders of a graph G_r , the extraction width of an arbitrary graph G_r is the minimum width of any extraction order: $\text{ew}(G_r) = \min_{G_r^{\mathcal{X}} \in \mathcal{X}(G_r)} \text{ew}_{\mathcal{X}}(G_r^{\mathcal{X}})$. ◻

Figure 3 depicts an example extraction order containing 5 confluences, which can be uniquely identified by their sources and targets: $C_{a,i}^{\mathcal{X}}, C_{i,j}^{\mathcal{X}}, C_{a,l}^{\mathcal{X}}, C_{b,l}^{\mathcal{X}}, C_{f,k}^{\mathcal{X}}$ with e.g. $C_{b,l}^{\mathcal{X}} = \{(b, i), (i, f), (f, l)\} \sqcup \{(b, d), (d, l)\}$. According to Definition 10, the edge bags of node f are $\mathcal{B}_{r,f}^{\mathcal{X}} = \{B_{r,f}^{\mathcal{X},1} = \{(f, j)\}, B_{r,f}^{\mathcal{X},2} = \{(f, g), (f, k)\}, B_{r,f}^{\mathcal{X},3} = \{(f, l)\}\}$ with the corresponding label sets being $\mathcal{L}_{r,f}^{\mathcal{X},1} = \{j\}$, $\mathcal{L}_{r,f}^{\mathcal{X},2} = \{k\}$, and $\mathcal{L}_{r,f}^{\mathcal{X},3} = \{l\}$. For node i , only a single edge bag $\mathcal{B}_{r,i}^{\mathcal{X}} = \{B_{r,i}^{\mathcal{X},1} = \{(i, c), (i, f)\}\}$ with label set $\mathcal{L}_{r,i}^{\mathcal{X},1} = \{j, l\}$ exists.

3.2 Structure of Edge Labels

In the following, we study the structure of extraction confluences and of the edge labels. We employ the following notation for indicating edges being reachable from and/or by nodes in the extraction order.

► **Definition 12** (Reachable Edge Sets). Given an extraction order $G_r^{\mathcal{X}}$, we denote by $E_{r,i}^{\mathcal{X},\text{suc}}, E_{r,j}^{\mathcal{X},\text{pre}} \subseteq E_r^{\mathcal{X}}$ the set of edges which can be reached from $i \in V_r$ and which may lead to $j \in V_r$. We denote by $E_{r,i \rightsquigarrow j}^{\mathcal{X}} = E_{r,i}^{\mathcal{X},\text{suc}} \cap E_{r,j}^{\mathcal{X},\text{pre}}$ the edges lying on a path from i to j . ◻

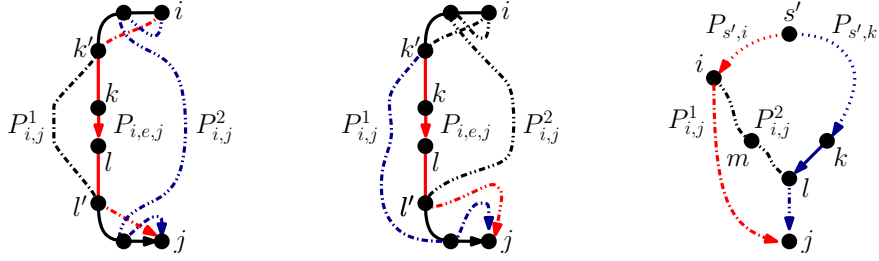
The following lemma forms the basis for efficiently computing edge labels.

► **Lemma 13.** *Edge $e \in E_r^{\mathcal{X}}$ is labeled with $j \in V_r$ iff. there exists a node $i \in V_r$, such that (i) e lies on a path from i to j , i.e. $e \in E_{r,i \rightsquigarrow j}^{\mathcal{X}}$, and (ii) a confluence $C_{i,j}^{\mathcal{X}}$ from i to j exists.*

Proof. It is easy to see that the above two conditions are necessary. Clearly, if the first condition does not hold for some node $i \in V_r$, then there cannot exist a confluence from i to j covering the edge e . Secondly, if there does not exist any confluence between i and j , then there cannot exist a confluence from i to j covering e .

We now show that these conditions are also sufficient. First, note that any path from i to j must be contained in $E_{r,i \rightsquigarrow j}^{\mathcal{X}}$. Let $e \in E_{r,i \rightsquigarrow j}^{\mathcal{X}}$ denote any edge for which the above conditions hold. We show that edge e lies on a confluence with target j . By the second condition, there exist two node-disjoint paths $P_{i,j}^1, P_{i,j}^2 \subseteq E_{r,i \rightsquigarrow j}^{\mathcal{X}}$ from i to j . Now, if e lies on either of these paths, then $P_{i,j}^1 \sqcup P_{i,j}^2$ already constitutes a confluence. Hence, assume that e does not lie on either of these paths. Let $e = (k, l)$, i.e. k is the tail and l the head. Furthermore, let $P_{i,k} \subseteq E_{r,i \rightsquigarrow j}^{\mathcal{X}}$ denote any path from i to k and denote by $P_{l,j} \subseteq E_{r,i \rightsquigarrow j}^{\mathcal{X}}$ any path from l to j . Let $P_{i,e,j}$ denote the path obtained from joining $P_{i,k}$, $e = (k, l)$, and $P_{l,j}$.

If $P_{i,e,j}$ only intersects with $P_{i,j}^1$ (or $P_{i,j}^2$), then $P_{i,e,j}$ together with $P_{i,j}^2$ (or $P_{i,j}^1$) constitutes a confluence towards j which covers e , proving our claim. Hence, assume that $P_{i,e,j}$ intersects with both paths. Let k' be the last node on path $P_{i,k}$ which also lies on $P_{i,j}^1$ or $P_{i,j}^2$ and let l' denote the first node on $P_{l,j}$ which also lies on $P_{i,j}^1$ or $P_{i,j}^2$. Assume now w.l.o.g. that both k' and l' lie on path $P_{i,j}^1$, then the subpath of $P_{i,j}^1$ from k' to l' can be substituted with the subpath from k' to l' of $P_{i,e,j}$, yielding the confluence depicted on the left of Figure 4. On the other hand, if k' lies on $P_{i,j}^1$ while l' lies on path $P_{i,j}^2$, then there exists a confluence from k' to j covering the edge e : using the suffix of path $P_{i,j}^1$ starting at node k' as first path and



■ **Figure 4** Visualizations of the constructions used in the proofs of Lemma 13 (left and center) and Lemma 15 (right) to show that an edge (k, l) is covered by a confluence. The confluence path $P_{i,j}^1$ is dashed with a single dot and the confluence path $P_{i,j}^2$ is dashed with two dots. The constructed confluences consist of the highlighted paths in red and blue.

Lemma 13 (left and center): Construction of a confluence when $P_{i,j}^1$ intersects with k' and l' (left). Construction of a confluence when k' lies on $P_{i,j}^1$, while l' lies on $P_{i,j}^2$ (center).

Lemma 15 (right): As there must exist a node r' reaching both i and m , a confluence with target j is constructed covering the edge (m, l) .

the subpath of $P_{i,e,k}$ from k' to l' together with the suffix of $P_{i,j}^2$ starting at l' , a confluence is found that covers e . By construction, as the nodes of path $P_{i,e,j}$ between k' and l' do neither lie on $P_{i,j}^1$ nor $P_{i,j}^2$, the paths of the constructed confluence are disjoint (see Figure 4 (center) for a visualization). Hence, the two conditions stated in the lemma are also sufficient to decide whether an edge $e \in E_r^{\mathcal{X}}$ is covered by a confluence towards $j \in V_r$ holds. ■

Based on Lemma 13, the edge labels can be computed in polynomial-time. Concretely we apply Menger's theorem [18] to decide for any combination of virtual nodes $i, j \in V_r$, whether two disjoint paths exist from i to j . If this is the case, then all edges lying in $E_{r,i \rightsquigarrow j}^{\mathcal{X}}$ must be labeled with j and we obtain:

► **Lemma 14.** *The edge labels $\mathcal{L}_{r,e}^{\mathcal{X}}$ can be computed in time $\mathcal{O}(|V_r|^3 \cdot |E_r|)$.*

Proof. We argue that the conditions of Lemma 13 can be checked in polynomial time. For each potential target node $j \in V_r$ and each source node $i \in V_r$, we check whether two node-disjoint paths exist from i to j by applying Menger's theorem [18]: for each node k lying on a path from i to j , we decide whether j is still reachable from i when k is removed. If this is true for each intermediate node, then by Menger's theorem, there exist at least two node-disjoint paths from i to j and hence there must exist a confluence $C_{i,j}^{\mathcal{X}}$. Hence, given the existence of a confluence, all edges in $E_{r,i \rightsquigarrow j}^{\mathcal{X}}$ are labeled by j . At most $\mathcal{O}(|V_r|^2)$ many node pairs need to be considered and the check whether two node-disjoint paths exist can be implemented in time $\mathcal{O}(|V_r| \cdot |E_r|)$. Hence, the overall runtime to compute all labels is bounded by $\mathcal{O}(|V_r|^3 \cdot |E_r|)$. ■

The two following lemmas state important structural properties for edge labels, namely that incoming edges are always labeled the same and that each label has a unique source.

► **Lemma 15.** *$\mathcal{L}_{r,e}^{\mathcal{X}} = \mathcal{L}_{r,e'}^{\mathcal{X}}$ holds for any pair of incoming edges $e, e' \in \delta_{\mathcal{X}}^-(l)$ of node $l \in V_r$.*

Proof. Assume for the sake of contradiction that an edge $e = (m, l)$ is labeled with j , i.e. $j \in \mathcal{L}_{r,e}^{\mathcal{X}}$, and that some other incoming edge $e' = (k, l)$ is not labeled with j , i.e. $j \notin \mathcal{L}_{r,e'}^{\mathcal{X}}$. As the edge e is labeled with j , there must exist some confluence $C_{i,j}^{\mathcal{X}}$ covering e . As the edge $e' = (k, l)$ is not labeled with j , we obtain from Lemma 13 that the edge e' is not reachable from i , i.e., $e' \notin E_{r,i \rightsquigarrow j}^{\mathcal{X}}$ holds. As both i and k are reachable from the root s_r of $G_r^{\mathcal{X}}$, there

must exist paths $P_{s_r,i}$ and $P_{s_r,k}$ leading from the root s_r to i and k , respectively. Let s' denote the last node that lies on both of these paths. The subpaths $P_{s',i}$ and $P_{s',k}$ of $P_{s_r,i}$ and $P_{s_r,k}$ starting at s' do not use any of the edges in $E_{r,i \rightsquigarrow j}^{\mathcal{X}}$. Hence, joining $P_{s',i}$ with $P_{i,j}^1$ and joining $P_{s',k}$ with $e' = (k,l)$ and the subpath of $P_{i,j}^2$ beginning at node l , a confluence is constructed that covers edge e' (see Figure 4 (right) for an visualization of the construction). Therefore, also e' must be labeled with j , yielding a contradiction to our assumption that e' was not labeled by j and all incoming edges must be labeled the same. ■

Lastly, the following lemma shows that any label is introduced only once.

► **Lemma 16.** *For each label $j \in V_r$ there exists a unique root node $s_j \in V_r$, such that:*

1. *Any edge $e \in E_r^{\mathcal{X}}$ being labeled with $j \in \mathcal{L}_{r,e}^{\mathcal{X}}$ is contained in $E_{r,s_j \rightsquigarrow j}^{\mathcal{X}}$.*
2. *Any path from s_r (the root of the extraction order) to j passes through s_j .*

Proof. Consider two nodes $i, i' \in V_r$ being the sources of confluences $C_{i,j}^{\mathcal{X}}$ and $C_{i',j}^{\mathcal{X}}$ towards j . Assume that neither i occurs in $E_{r,i' \rightsquigarrow j}^{\mathcal{X}}$ nor that i' occurs in $E_{r,i \rightsquigarrow j}^{\mathcal{X}}$. As the graph $G_r^{\mathcal{X}}$ is rooted, there must exist a node s' reaching both i and i' and spawning a confluence towards j . Furthermore, $(E_{r,i \rightsquigarrow j}^{\mathcal{X}} \cup E_{r,i' \rightsquigarrow j}^{\mathcal{X}}) \subseteq E_{r,s' \rightsquigarrow j}^{\mathcal{X}}$ holds in this case. Hence, for any pair of nodes i, i' being sources of confluences towards j , either one of the nodes is reachable from the other, or there exists another node $s' \in V_r$ such that $E_{r,i \rightsquigarrow j}^{\mathcal{X}}$ and $E_{r,i' \rightsquigarrow j}^{\mathcal{X}}$ are contained in $E_{r,s' \rightsquigarrow j}^{\mathcal{X}}$. Clearly, as either i dominates i' or i' dominates i , or there exists some other node s' dominating both, there must exist a single unique root node $s_j \in V_r$ such that all edges labeled with j lie in $E_{r,s_j \rightsquigarrow j}^{\mathcal{X}}$.

The second claim is immediate: if there was to exist some path from the root s_r to an edge being labeled with j without passing through the unique root $s_j \in V_r$, then there must exist a confluence $C_{s',j}^{\mathcal{X}}$ starting at some other node $s' \in V_r$, such that s' reaches s_j but s_j does not reach s' . Hence, by our above observation s' dominates s_j and s_j cannot be the unique root. Thus, all paths from the root must pass through s_j on their way to j . ■

3.3 Novel Decomposable LP Formulation

Our novel Linear Programming Formulation 3 is based on the idea to decide the mapping locations of confluence targets a priori. We do so by considering copies or sub LPs of the MCF formulation (see Constraint 13) and we employ the following notation. For an edge $e = (i, j) \in E_r$, we denote by $G_{r,e} = (V_{r,e}, E_{r,e})$ with $V_{r,e} = \{i, j\}$ and $E_{r,e} = \{e\}$ the subgraph of G_r containing only edge e . Variables of sub LPs are named as before, but are now additionally indexed: $\alpha[\beta]$ denotes the variable α in the copy identified by β . To denote the combinations of mapping possibilities of labels, we employ $\mathcal{M}(X)$ to denote the function space from the set X to V_S , i.e., $\mathcal{M}(X) = [X \rightarrow V_S]$. Accordingly, considering an edge $e \in E_r$ of request $r \in \mathcal{R}$ being labeled by $\mathcal{L}_{r,e}$, we instantiate one copy of the MCF formulation per edge label mapping $m_e^{\mathcal{L}} \in \mathcal{M}(\mathcal{L}_{r,e})$ (cf. Constraint 13). For better readability, we write $\langle f|Z \rangle : Z \rightarrow Y$ to denote $f|_Z$, i.e., the (standard) restriction of the function $f : X \rightarrow Y$ on the subset $Z \subseteq X$. Hence, $\langle f|Z \rangle(z) = f(z)$ holds for $z \in Z$.

To link the LP copies, we employ two types of *global* node mapping variables. We use the (global) $y_{r,i}^u$ variables already presented in Formulation 2 as well as node mapping variables $\gamma_{r,i,b,a}^u \in [0, 1]$ for *edge bags* $B_{r,i}^{\mathcal{X},b} \in \mathcal{B}_{r,i}^{\mathcal{X}}$, each mapping $m_a^{\mathcal{L}} \in \mathcal{M}(\mathcal{L}_{r,i}^{\mathcal{X},b})$ of the labels contained in the respective edge bag, and the mapping locations $u \in V_S^{r,i}$. The classic variables $y_{r,i}^u$ are used for coupling the embedding variable x_r and the sub-LP node mapping variables (see Constraints 14 and 15) as well as for computing the node load (see Constraint 19). The

Formulation 3: Novel Decomposable Base Formulation for the VNEP

$$(6) - (10) \text{ for } G_{r,e} \text{ on variables } (x_r, \vec{y}_r, \vec{z}_r, \vec{a}_r) \llbracket e, m_e^{\mathcal{L}} \rrbracket \quad \forall r \in \mathcal{R}, e \in E_r, m_e^{\mathcal{L}} \in \mathcal{M}(\mathcal{L}_{r,e}) \quad (13)$$

$$x_r = \sum_{u \in V_S^{r,i}} y_{r,i}^u \quad \forall r \in \mathcal{R} \quad (14)$$

$$y_{r,i}^u = \sum_{m_e^{\mathcal{L}} \in \mathcal{M}(\mathcal{L}_{r,e})} y_{r,i}^u \llbracket e, m_e^{\mathcal{L}} \rrbracket \quad \forall r \in \mathcal{R}, i \in V_r, u \in V_S^{r,i}, e \in E_r : i \in V_{r,e} \quad (15)$$

$$y_{r,i}^u \llbracket \vec{E}_r(e), m_e^{\mathcal{L}} \rrbracket = \sum_{\substack{m_a^{\mathcal{L}} \in \mathcal{M}(\mathcal{L}_{r,i}^{\mathcal{X},b}): \\ \langle m_a^{\mathcal{L}} | \mathcal{L}_{b \cap e}^{\mathcal{X}} \rangle = m_e^{\mathcal{L}}}} \gamma_{r,i,b,a}^u \quad \forall r \in \mathcal{R}, i \in V_r, u \in V_S^{r,i}, B_{r,i}^{\mathcal{X},b} \in \mathcal{B}_{r,i}^{\mathcal{X}}, \\ e \in B_{r,i}^{\mathcal{X},b}, m_e^{\mathcal{L}} \in \mathcal{M}(\mathcal{L}_{r,e}^{\mathcal{X}}) \quad (16)$$

$$\sum_{\substack{m_e^{\mathcal{L}} \in \mathcal{M}(\mathcal{L}_{r,e}^{\mathcal{X}}): \\ \langle m_e^{\mathcal{L}} | \mathcal{L}_{b \cap e}^{\mathcal{X}} \rangle = m_{b \cap e}^{\mathcal{L}}}} y_{r,i}^u \llbracket \vec{E}_r(e), m_e^{\mathcal{L}} \rrbracket = \sum_{\substack{m_a^{\mathcal{L}} \in \mathcal{M}(\mathcal{L}_{r,i}^{\mathcal{X},b}): \\ \langle m_a^{\mathcal{L}} | \mathcal{L}_{b \cap e}^{\mathcal{X}} \rangle = m_{b \cap e}^{\mathcal{L}}}} \gamma_{r,i,b,a}^u \quad \forall r \in \mathcal{R}, i \in V_r, u \in V_S^{r,i}, e \in \delta_{\vec{\mathcal{X}}}^-(i), \\ B_{r,i}^{\mathcal{X},b} \in \mathcal{B}_{r,i}^{\mathcal{X}}, m_{b \cap e}^{\mathcal{L}} \in \mathcal{M}(\mathcal{L}_{b \cap e}^{\mathcal{X}}) \quad (17)$$

$$y_{r,i}^u \llbracket \vec{E}_r(e), m_e^{\mathcal{L}} \rrbracket = 0 \quad \forall r \in \mathcal{R}, i \in V_r, e \in \delta_{\vec{\mathcal{X}}}^-(i) : i \in \mathcal{L}_{r,e}^{\mathcal{X}}, \\ m_e^{\mathcal{L}} \in \mathcal{M}(\mathcal{L}_{r,e}^{\mathcal{X}}), u \in V_S^{r,i} \setminus \{m_e^{\mathcal{L}}(i)\} \quad (18)$$

$$a_r^{\tau,u} = \sum_{i \in V_r, \tau_r(i)=\tau} d_r(i) \cdot y_{r,i}^u \quad \forall r \in \mathcal{R}, (\tau, u) \in R_S^V \quad (19)$$

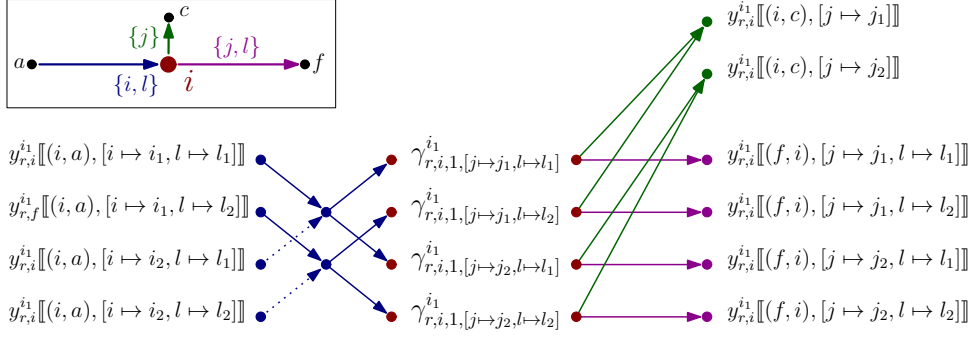
$$a_r^{u,v} = \sum_{\substack{e \in E_r \\ m_e^{\mathcal{L}} \in \mathcal{M}(\mathcal{L}_{r,e})}} a_r^{u,v} \llbracket e, m_e^{\mathcal{L}} \rrbracket \quad \forall r \in \mathcal{R}, (u, v) \in E_S \quad (20)$$

$$\sum_{r \in \mathcal{R}} a_r^{x,y} \leq d_S(x, y) \quad \forall (x, y) \in R_S \quad (21)$$

$$x_r \in [0, 1], \forall r \in \mathcal{R}; \quad y_{r,i}^u \in [0, 1], \forall r \in \mathcal{R}, i \in V_r, u \in V_S^{r,i}; \quad a_r^{x,y} \geq 0, \forall r \in \mathcal{R}, (x, y) \in R_S \\ \gamma_{r,i,b,a}^u \in [0, 1], \forall r \in \mathcal{R}, i \in V_r, u \in V_S^{r,i}, B_{r,i}^{\mathcal{X},b} \in \mathcal{B}_{r,i}^{\mathcal{X}}, m_a^{\mathcal{L}} \in \mathcal{M}(\mathcal{L}_{r,i}^{\mathcal{X},b}) \quad (22)$$

node mapping variables for edge bags $\gamma_{r,i,b,a}^u$ are defined for all mappings of their label set $m_a^{\mathcal{L}} \in \mathcal{M}(\mathcal{L}_{r,i}^{\mathcal{X},b})$. As $\mathcal{L}_{r,e}^{\mathcal{X}} \subseteq \mathcal{L}_{r,i}^{\mathcal{X},b}$ holds for all edges $e \in B_{r,i}^{\mathcal{X},b}$, the node mapping variables of an edge bag directly induce node mappings for all edges contained in the respective bag (see Constraint 16).

In the following, we argue how ‘flows’ are induced and accordingly how solutions to the formulation can be decomposed. Figure 5 visualizes the workings of Constraints 16 to 18. Assuming that $x_r > 0$ holds, then by Constraint 14 there will exist a substrate node $u \in V_S^{r,s_r}$ onto which the root s_r is placed, i.e. $y_{r,s_r}^u > 0$ holds. Constraint 15 distributes the quantity y_{r,s_r}^u over the sub LP node mapping variables while Constraint 16 ensures that these node mapping variables agree with each other. Due to the validity of the MCF Formulation 2, by setting the node mapping variable for one of the endpoints of the edge graph $G_{r,e}$, the node mapping variables of the other endpoint of $G_{r,e}$ must be set accordingly. On the other hand, Constraint 17 ensures that any incoming edge (according to the extraction order) agrees with the respective node bag variables and hence force the further distribution of ‘flows’. The correctness of the formulation then mostly follows from the following observations: (i) Based on the acyclicity of the extraction order and the fact that all nodes can be reached from the root s_r , ‘flow’ is distributed throughout the whole request graph. (ii) A novel edge label j is introduced only exactly once according to Lemma 16, namely at the root $s_j \in V_r$ and hence only at node s_j the a priori mapping of j is fixed. (iii) For any confluence $C_{k,i}^{\mathcal{X}}$ with target i all incoming edges of i are itself labeled by i (cf. Lemma 15). Accordingly, considering a mapping $m_e^{\mathcal{L}} \in \mathcal{M}(\mathcal{L}_{r,e})$ of an incoming edge $e \in \delta_{\vec{\mathcal{X}}}^-(i)$, Constraint 18 explicitly forbids node placements of i to nodes $V_S^{r,i} \setminus \{m_e^{\mathcal{L}}(i)\}$ in the respective sub LPs. Hence, incoming edges of node i labeled by $m_e^{\mathcal{L}}$ can only map i to $m_e^{\mathcal{L}}(i) \in V_S^{r,i}$.



■ **Figure 5** Visualization of the relation of the different node mapping variables for the example of Figure 3 under the assumption that the virtual nodes $i, j, k \in V_r$ can be mapped only on $i_k, j_k, l_k \in V_S$ for $k \in \{1, 2\}$, respectively. Depicted are only the variables relating to the mapping of i to i_1 . To highlight that the LP copies are created upon the original orientation of edges, we assume that $(i, a), (f, i) \in E_r$ holds and that these were reversed for G_r^X depicted in Figure 3 on Page 10. The edges and nodes are to be read as ‘flows’ for which flow preservation holds. The directions of the edges shall help the reader to follow how the node mapping variables of ‘incoming’ edges trigger the node mapping variables of ‘outgoing’ edges. The connections on the left are due to Constraint 17 and the connections on the right are due to Constraint 16. Note that the dashed edges on the left will be 0 due to Constraint 18: in the index of the respective sub LP, the virtual node i is mapped onto i_2 and hence the respective dashed variables are set to 0.

► **Theorem 17.** *Considering specific extraction orders G_r^X for each request $r \in \mathcal{R}$, the size of the novel decomposable Formulation 3 is bounded by $\mathcal{O}(\sum_{r \in \mathcal{R}} |G_S|^{\text{ew}_X(G_r^X)} \cdot |G_r|)$.*

Proof. Consider a single request $r \in \mathcal{R}$ and a fixed extraction order G_r^X . There are at most $\text{ew}_X(G_r^X) - 1$ many sub LPs for each edge $e \in E_r^X$, as $|\mathcal{L}_{r,e}^X| \leq \text{ew}_X(G_r^X) - 1$ holds by definition. Otherwise, the formulation’s size is dominated by the node bag mapping variables $\gamma_{r,i,b,a}^u$ and the respective Constraints 16 - 18. Since the bags partition the outgoing edges and encode all potential mappings of the respective label set $\mathcal{L}_{r,i}^{X,b}$ while including the mapping location of the respective virtual node $i \in V_r$, the size of the respective formulation parts is bounded by $\mathcal{O}(|V_r| \cdot |V_S|^{\text{ew}_X(G_r^X)})$. The result is obtained by summing over the requests. ■

3.4 Decomposition Algorithm for the Novel LP Formulation

We now formally present the decomposition algorithm (see Algorithm 2) and prove its correctness. The algorithm builds on the ideas of the decomposition algorithm for the MCF Formulation 2 presented in Section 2.2.

Fixing the mapping of the root initially, mappings for the outgoing edges (with respect to the extraction order) are extracted again together with the mappings of the heads of these edges using Lemma 4. However, as the edge embeddings are computed using a copy of the MCF formulation for each node mapping function of the edge’s labels, the mapping of the edge’s labels to substrate nodes must be fixed first. To this end, we employ the node mapping variables $\gamma_{r,i,b,a}^u$ of the edge bags. Concretely, whenever the outgoing edges of $i \in V_r$ are mapped, we show that given the mapping of the virtual node i onto some substrate node $m_r^V(i) = u$, we can always find a variable $\gamma_{r,i,b,a}^u$ for edge bag $B_{r,i}^{X,b} \in \mathcal{B}_{r,i}^X$ and $m_a^L \in \mathcal{M}(\mathcal{L}_{r,i}^{X,b})$, such that (i) the mapping of the bag m_a^L agrees with the previous node mappings, i.e., $m_a^L(i') = m_r^V(i')$ holds for all previously mapped virtual nodes i' , and that (ii) the variable $\gamma_{r,i,b,a}^u$ is strictly greater 0. Given such a variable $\gamma_{r,i,b,a}^u$ and fixing the

Algorithm 2: Decomposition algorithm for solutions to the novel LP formulation 3

Input : Request $r \in \mathcal{R}$, extraction order $G_r^{\mathcal{X}}$, solution to Formulation 3

Output : Convex combination $\mathcal{D}_r = \{D_r^k = (f_r^k, m_r^k)\}_k$ of valid mappings

```

1  set  $\mathcal{D}_r \leftarrow \emptyset$  and  $k \leftarrow 1$ 
2  while  $x_r > 0$  do
3    set  $m_r^k = (m_r^V, m_r^E) \leftarrow (\emptyset, \emptyset)$ 
4    set  $\mathcal{Q} \leftarrow \{s_r\}$ 
5    choose  $u \in V_S^{r,s_r}$  with  $y_{r,s_r}^u > 0$  and set  $m_r^V(s_r) \leftarrow u$ 
6    while  $|\mathcal{Q}| > 0$  do
7      choose  $i \in \mathcal{Q}$  and set  $\mathcal{Q} \leftarrow \mathcal{Q} \setminus \{i\}$ 
8      foreach  $B_{r,i}^{\mathcal{X},b} \in \mathcal{B}_{r,i}^{\mathcal{X}}$  do
9        let  $M^V = (m_r^V)^{-1}(V_S)$  denote the already mapped nodes
10       choose  $m_a^{\mathcal{L}} \in \mathcal{M}(\mathcal{L}_{r,i}^{\mathcal{X},b})$ , s.t.  $\gamma_{r,i,b,a}^{m_r^V(i)} > 0$ 
11       and  $\langle m_a^{\mathcal{L}} | \mathcal{L}_{r,i}^{\mathcal{X},b} \cap M^V \rangle = \langle m_r^V | \mathcal{L}_{r,i}^{\mathcal{X},b} \cap M^V \rangle$ 
12       set  $m_r^V(j) \leftarrow m_a^{\mathcal{L}}(j)$  for all  $j \in \mathcal{L}_{r,i}^{\mathcal{X},b} \setminus M^V$ 
13       foreach  $e = (i, j) \in B_{r,i}^{\mathcal{X},b}$  do
14         if  $(i, j) = \vec{E}_r(i, j)$  then
15           compute path  $P_{r,i,j}^{u,v}$  from  $m_r^V(i) = u$  to  $v \in V_S^{r,j}$  according to Lemma 4
16           s.t.  $y_{r,j}^v \llbracket (i, j), \langle m_r^V | \mathcal{L}_{r,e} \rangle \rrbracket > 0$  and
17            $z_{r,i,j}^{u',v'} \llbracket (i, j), \langle m_r^V | \mathcal{L}_{r,e} \rangle \rrbracket > 0$  for all  $(u', v') \in P_{r,i,j}^{u,v}$ 
18           set  $m_r^E(i, j) \leftarrow P_{r,i,j}^{u,v}$  and if  $m_r^V(j) = \emptyset$  then  $m_r^V(j) \leftarrow v$ 
19         else
20           compute path  $P_{r,j,i}^{v,u}$  from  $v \in V_S^{r,j}$  to  $m_r^V(i) = u$  according to Lemma 4
21           s.t.  $y_{r,j}^v \llbracket \vec{E}_r(i, j), \langle m_r^V | \mathcal{L}_{r,e} \rangle \rrbracket > 0$  and
22            $z_{r,j,i}^{u',v'} \llbracket \vec{E}_r(i, j), \langle m_r^V | \mathcal{L}_{r,e} \rangle \rrbracket > 0$  for all  $(u', v') \in P_{r,j,i}^{v,u}$ 
23           set  $m_r^E(\vec{E}_r(i, j)) \leftarrow P_{r,j,i}^{v,u}$  and if  $m_r^V(j) = \emptyset$  then  $m_r^V(j) \leftarrow v$ 
24         if  $m_r^E(\vec{E}_r(e)) \neq \emptyset$  for all  $e \in \delta_{\mathcal{X}}^-(j)$  then
25           set  $\mathcal{Q} \leftarrow \mathcal{Q} \cup \{j\}$ 
26
27       set  $\mathcal{V}_k \leftarrow \left( \begin{array}{l} \{x_r\} \cup \{y_{r,i}^u \mid i \in V_r, u = m_r^V(i)\} \\ \cup \{x_r \llbracket e, \langle m_r^V | \mathcal{L}_{r,e} \rangle \rrbracket \mid e \in E_r\} \\ \cup \{y_{r,i}^u \llbracket e, \langle m_r^V | \mathcal{L}_{r,e} \rangle \rrbracket \mid e \in E_r, i \in V_{r,e}, u = m_r^V(i)\} \\ \cup \{z_{r,i,j}^{u,v} \llbracket e, \langle m_r^V | \mathcal{L}_{r,e} \rangle \rrbracket \mid e = (i, j) \in E_r, (u, v) \in m_r^E(i, j)\} \\ \cup \{\gamma_{r,i,b,a}^u \mid i \in V_r, u = m_r^V(i), B_{r,i}^{\mathcal{X},b} \in \mathcal{B}_{r,i}^{\mathcal{X}}, m_a^{\mathcal{L}} = \langle m_r^V | \mathcal{L}_{r,i}^{\mathcal{X},b} \rangle\} \end{array} \right)$ 
28       set  $f_r^k \leftarrow \min \mathcal{V}$ 
29       set  $v \leftarrow v - f_r^k$  for all  $v \in \mathcal{V}$ 
30       set  $a_r^{x,y} \leftarrow a_r^{x,y} - f_r^k \cdot A(m_r^k, x, y)$  for all  $(x, y) \in R_S$ 
31       foreach  $(i, j) \in E_r$  and each  $(x, y) \in \{(\tau_r(i), i), (\tau_r(j), j), (i, j)\}$  do
32         set  $a_r^{x,y} \llbracket (i, j), \langle m_r^V | \mathcal{L}_{r,e} \rangle \rrbracket \leftarrow a_r^{x,y} \llbracket \langle m_r^V | \mathcal{L}_{r,e} \rangle \rrbracket - f_r^k \cdot A(m_r^k, x, y)$ 
33       add  $D_r^k = (f_r^k, m_r^k)$  to  $\mathcal{D}_r$  and set  $k \leftarrow k + 1$ 
34  return  $\mathcal{D}_r$ 

```

node mappings of all the new labels contained in $\mathcal{L}_{r,e}^{\mathcal{X}}$ according to $m_a^{\mathcal{L}}$, mappings for the outgoing edges can always be extracted due to Constraint 16. Concretely, after having fixed the mappings of the labels of the outgoing edge e , Constraint 16 ensures that for the sub LP with index $\langle m_r^V | \mathcal{L}_{r,e}^{\mathcal{X}} \rangle$ the condition $y_{r,i}^u > 0$ holds, thereby allowing the application of Lemma 4 to extract the mapping for the edge e . Having given this intuition, we now formally prove its correctness.

► **Theorem 18.** *For a request $r \in \mathcal{R}$ and its extraction order $G_r^{\mathcal{X}}$, a solution to Formulation 3 can be decomposed into $\mathcal{D}_r = \{(f_r^k, m_r^k)\}_k$ in time $\mathcal{O}(|G_S|^{2 \cdot \text{ew}_X(G_r^{\mathcal{X})} + 1} \cdot |G_r|^2)$, such that:*

- *The decomposition is complete, i.e., $x_r = \sum_k f_r^k$ holds.*
- *Allocations are bounded by \vec{a}_r , i.e., $a_r^{x,y} \geq \sum_{(f_r^k, m_r^k)} f_r^k \cdot A(m_r^k, x, y)$ holds for $(x, y) \in R_S$.*

Proof. We prove that each iteration yields a valid mapping m_r^k of value $f_r^k > 0$.

First, note that if in Line 10 a suitable mapping $m_a^{\mathcal{L}}$ was found, such that the respective edge bag variable $\gamma_{r,i,b,a}^{m_r^V(i)}$ is positive, then the requirement of Lemma 4 that $y_{r,i}^{m_r^V(i)} \llbracket e, \langle m_r^V | \mathcal{L}_{r,e}^{\mathcal{X}} \rangle \rrbracket > 0$ holds is always satisfied due to Constraint 16.

The initial mapping of the root in Line 5 is always possible due to Constraints 14. Furthermore, when considering the edge bags of the root s_r , there will always exist a suitable edge bag variable $\gamma_{r,s_r,b,a}^{m_r^V(s_r)} > 0$ to choose from due to Constraints 15 and 16.

Having chosen a suitable mapping for the labels of the edge bag, the extraction of mappings for the outgoing edges $e \in B_{r,i}^{\mathcal{X},b}$ is always feasible, as Constraint 16 induces $y_{r,s_r}^{m_r^V(s_r)} > 0$ in the respective sub LPs. Also note that the application of Lemma 4 safeguards that the head j is mapped positively on some substrate node i , i.e. we have $y_{r,j}^v \llbracket \langle m_r^V | \mathcal{L}_{r,e}^{\mathcal{X}} \rangle \rrbracket > 0$.

Given the initial validity of the mapping of the root and its outgoing edges, assume now for the sake of contradiction that the extraction process fails at some point in time. Concretely, we consider the first point in time at which the constructed (partial) mapping $m_r^k = (m_r^V, m_r^E)$ is not valid anymore or at which the choose operation in Line 10 fails.

We first consider the case that the mapping m_r^k is not valid (anymore), such that the mapping of an edge $e = (i, j)$ fails to start at $m_r^V(i)$ or fails to lead to $m_r^V(j)$. Edges are only mapped in Lines 15 and 18 and we consider w.l.o.g. that the algorithm fails in Line 15. For this type of failure to happen, the node j must have been mapped before as the node j is otherwise validly mapped by the same line of the pseudocode. As j can only be mapped multiple times if j is itself a label and all incoming edges of a node share the same labels (see Lemma 15), the edge (i, j) must have been labeled by j . Let v' denote the substrate node in which the path $m_r^E(i, j)$ ends and for which $v' \neq m_r^V(i)$ holds. As stated in the beginning of the proof, the requirement of Lemma 4 is always valid (if a suitable mapping $m_a^{\mathcal{L}}$ was found) and hence, by applying Lemma 4 we obtain $y_{r,j}^{v'} \llbracket e, \langle m_r^V | \mathcal{L}_{r,e}^{\mathcal{X}} \rangle \rrbracket > 0$. However, Constraint 18 clearly forbids the use of this node v' as it does not equal $m_r^V(i)$ by setting $y_{r,j}^{v'} \llbracket e, \langle m_r^V | \mathcal{L}_{r,e}^{\mathcal{X}} \rangle \rrbracket = 0$. This is a contradiction, and the only option for the extraction process to fail is hence due to an infeasible choose operation in Line 10.

As argued in the beginning of the proof, the choose operation may not fail for the root. Hence, the node $i \in V_r$ for which Line 10 fails, is not the root and has been reached by at least one incoming edge $(k, i) \in E_r^{\mathcal{X}}$. Assume that the choose operation fails for a specific edge bag $B_{r,i}^{\mathcal{X},b} \in \mathcal{B}_{r,i}^{\mathcal{X}}$.

We first show that $\mathcal{L}_{r,i}^{\mathcal{X},b} \cap M^V \subseteq \mathcal{L}_{r,(k,i)}^{\mathcal{X}}$ holds. Assume for the sake of contradiction, that there exists some label $l \in \mathcal{L}_{r,i}^{\mathcal{X},b} \cap M^V$ such that $l \notin \mathcal{L}_{r,(k,i)}^{\mathcal{X}}$. As the label l is contained in M^V , a mapping was decided for l at some other node $s_l \in V_r$. In particular, Lemma 16 specifies that the node s_l is the unique root of all the confluences towards l , such that any other node

with an edge being labeled by l must be reachable from s_l . However, as all incoming labels agree on their labels (see Lemma 15) and no incoming edge of the node i hence lies on a confluence with target l , we must have $i = s_l$. In this case however, the node mapping of l cannot have been decided before as the algorithm only fixes these node mappings once the choose operation was executed at the respective node s_l . Hence, $\mathcal{L}_{r,i}^{\mathcal{X},b} \cap M^V \subseteq \mathcal{L}_{r,(k,i)}^{\mathcal{X}}$ holds.

As all previous mapping steps have been valid and the mappings were obtained by the application of Lemma 4, we know that $y_{r,i}^u \llbracket \vec{E}_r((k,i)), \langle m_r^V | \mathcal{L}_{r,(k,i)}^{\mathcal{X}} \rangle \rrbracket > 0$ holds for $u = m_r^V(i)$. Consider now the particular mapping $m_{(k,i)}^{\mathcal{L}} = \langle m_r^V | \mathcal{L}_{b \cap (k,i)}^{\mathcal{X}} \rangle$ which is well-defined, as all labels of the incoming edge (k,i) must have been fixed before extracting the mapping of this edge. From the validity of Constraint 17 we obtain that $\sum_{m_a^{\mathcal{L}} \in \mathcal{M}(\mathcal{L}_{r,i}^{\mathcal{X},b}) : \langle m_a^{\mathcal{L}} | \mathcal{L}_{b \cap e}^{\mathcal{X}} \rangle = m_{(k,i)}^{\mathcal{L}}} \gamma_{r,i,b,a}^u > 0$ holds, as $y_{r,i}^u \llbracket \vec{E}_r((k,i)), \langle m_r^V | \mathcal{L}_{r,(k,i)}^{\mathcal{X}} \rangle \rrbracket$ is larger than 0. As $\mathcal{L}_{r,i}^{\mathcal{X},b} \cap M^V \subseteq \mathcal{L}_{r,(k,i)}^{\mathcal{X}}$ holds, we know that $\sum_{m_a^{\mathcal{L}} \in \mathcal{M}(\mathcal{L}_{r,i}^{\mathcal{X},b}) : \langle m_a^{\mathcal{L}} | \mathcal{L}_{r,i}^{\mathcal{X},b} \cap M^V \rangle = \langle m_r^V | \mathcal{L}_{r,i}^{\mathcal{X},b} \cap M^V \rangle} \gamma_{r,i,b,a}^u > 0$ holds, since the restriction in the sum's index has been loosened. Hence, the choose operation in Line 10 can always be successfully executed and the mapping constructed in the k -th iteration will always be valid.

It is easy to check that the claims with respect to the completeness and the bounds by the load variables also hold: the mapping is always covered by respective mapping variables in \mathcal{V}_k and as the load is computed as a function of these mapping variables, the extracted fractional resource allocations are also bounded by \vec{a}_r .

Lastly, every time a valid mapping is extracted, a mapping variable's value is set to 0. As the formulation has size $\mathcal{O}(|G_S|^{\text{ew}_X(G_r^{\mathcal{X}})} \cdot |G_r|)$ (cf. Theorem 17) for request r , and this also bounds the number of variables, at most $\mathcal{O}(|G_S|^{\text{ew}_X(G_r^{\mathcal{X}})} \cdot |G_r|)$ many valid mappings may be recovered. For recovering a single valid mapping, the runtime can be bounded by $\mathcal{O}(|G_S|^{\text{ew}_X(G_r^{\mathcal{X}})+1} \cdot |G_r|)$, as the **choose** operation in Line 10 is executed at most $|V_r|$ times and the path computations in Lines 14 and 17 can be implemented in time $\mathcal{O}(|E_S|)$ (cf. Lemma 4). Hence, the overall runtime of the decomposition algorithm is bounded by $\mathcal{O}(|G_S|^{2 \cdot \text{ew}_X(G_r^{\mathcal{X}})+1} \cdot |G_r|^2)$. ■

4 FPT-Approximations for the Virtual Network Embedding Problem

As shown above, our novel LP Formulation 3 is sufficiently strong, such that its solutions can be decomposed into convex combinations $\mathcal{D}_r = \{(f_r^k, m_r)\}_k$ for each request $r \in \mathcal{R}$. In this section we now apply randomized rounding [23] on the decomposed solution to obtain fixed-parameter tractable tri-criteria approximations for the profit and the cost variant of the VNEP.

In the following, we cast the quality of the found solutions in terms of random variables to bound the respective probabilities of not finding a suitable solution. To this end, we employ the following well-known tail bounds.

► **Theorem 19** (Chernoff-Bound [9]). *Let $X = \sum_{i=1}^n X_i$ be a sum of n independent random variables $X_i \in [0, 1]$. Then $\mathbb{P}(X \leq (1 - \varepsilon) \cdot \mathbb{E}(X)) \leq \exp(-\varepsilon^2 \cdot \mathbb{E}(X)/2)$ holds for $0 < \varepsilon < 1$.*

► **Theorem 20** (Hoeffding's Inequality [9]). *Given independent random variables $\{X_i\}_i$, s.t. $X_i \in [a_i, b_i]$, then $\mathbb{P}\left(\sum_i X_i - \mathbb{E}(\sum_i X_i) \geq t\right) \leq \exp(-2t^2 / (\sum_i (b_i - a_i)^2))$ holds.*

4.1 Approximating the Profit Variant

The pseudo-code of our approximation for the profit is presented as Algorithm 3. The algorithm first performs preprocessing in Lines 1-3 by removing all requests which cannot

Algorithm 3: Randomized Rounding Algorithm for the VNEP (Profit)

```

1 foreach  $r \in \mathcal{R}$  do // preprocess requests
2   compute LP solution to Formulation 3 for the request set  $\{r\}$  maximizing  $x_r$ 
3   remove request  $r$  from the set  $\mathcal{R}$  if  $x_r < 1$  holds
4 compute LP solution to Formulation 3 for request set  $\mathcal{R}$  maximizing  $\sum_{r \in \mathcal{R}} b_r \cdot x_r$ 
5 decompose LP solution into convex combinations  $\mathcal{D}_r$  for all  $r \in \mathcal{R}$ 
6 do // perform randomized rounding
7   construct solution by choosing mapping  $m_r^k$  with probability  $f_r^k$  for all  $r \in \mathcal{R}$ 
8 while the solution is not  $(\alpha, \beta, \gamma)$ -approximate and maximal rounding tries are not exceeded;

```

be fully (fractionally) embedded in the absence of other requests. As these requests cannot be fully embedded, these requests can never be part of any feasible solution and can hence be removed. In Lines 4-8 the randomized rounding scheme is applied: an LP solution to the Formulation 3 is computed, decomposed and then rounded. The rounding procedure is iterated as long as the constructed solution is not of sufficient quality or until the number of maximal rounding tries is exceeded. Concretely, we seek (α, β, γ) -approximate solutions which achieve at least a factor of $\alpha < 1$ times the optimal (LP) profit and exceed node and edge capacities by at most factors of $\beta \geq 1$ and $\gamma \geq 1$, respectively. In the following we discuss the parameters α , β , and γ for which solutions can be found *with high probability*.

Bounding the Profit. Employing the *discrete* random variable $Y_r \in \{0, b_r\}$ to model the profit achieved by (potentially) embedding request $r \in \mathcal{R}$, we have $\mathbb{P}(Y_r = b_r) = \sum_{(f_r^k, m_r^k) \in \mathcal{D}_r} f_r^k$ and $\mathbb{P}(Y_r = 0) = 1 - \sum_{(f_r^k, m_r^k) \in \mathcal{D}_r} f_r^k$. Hence, the overall profit achieved is $B = \sum_{r \in \mathcal{R}} Y_r$ with $\mathbb{E}(B) = \sum_{r \in \mathcal{R}} b_r \cdot \sum_{(f_r^k, m_r^k) \in \mathcal{D}_r} f_r^k$. As the decomposition is complete (cf. Theorem 18) we have $B_{\text{LP}} = \mathbb{E}(B)$, where B_{LP} denotes the profit of the LP solution.

By removing any request, which cannot be fully fractionally embedded (in the absence of other requests) in the preprocessing step, we know that $B_{\text{LP}} \geq \max_{r \in \mathcal{R}} b_r$ holds. By applying the Chernoff-Bound of Theorem 19, we obtain:

► **Lemma 21.** *The probability of achieving less than 1/3 of the profit of the optimal solution is upper bounded by $\exp(-2/9) \approx 0.8007$.*

Proof. Let $\hat{b} = \max_{r \in \mathcal{R}} b_r$ denote the maximum benefit of the requests. We consider the random variables $Y'_r = Y_r / \hat{b}$, such that $Y'_r \in [0, 1]$ holds. Let $B' = \sum_{r \in \mathcal{R}} Y'_r$ denote the total profit achieved after scaling down the profits. As $\mathbb{E}(B) = B_{\text{LP}} \geq \hat{b}$ holds, we have $\mathbb{E}(B') \geq 1$. Choosing $\varepsilon = 2/3$ and applying Theorem 19 on B' we obtain $\mathbb{P}(B' \leq (1/3) \cdot \mathbb{E}(B')) \leq \exp(-2 \cdot \mathbb{E}(B') / 9)$. Plugging in the *minimal* value of $\mathbb{E}(B')$, i.e., 1, into the equation we obtain: $\mathbb{P}(B' \leq (1/3) \cdot \mathbb{E}(B')) \leq \exp(-2/9)$ and accordingly $\mathbb{P}(B \leq (1/3) \cdot \mathbb{E}(B)) \leq \exp(-2/9)$.

As mentioned before, by Theorem 18 we have $\mathbb{E}(B) = B_{\text{LP}}$. Denoting the optimal profit of the Integer Program by B_{opt} and observing that $B_{\text{opt}} \leq B_{\text{LP}}$ holds as the IP is contained in the solution space of the LP, we have $B_{\text{opt}}/3 \leq B_{\text{LP}}/3 = \mathbb{E}(B)/3$. Hence, we obtain $\mathbb{P}(B \leq (1/3) \cdot B_{\text{opt}}) \leq \mathbb{P}(B \leq (1/3) \cdot \mathbb{E}(B)/3) \leq \exp(-2/9)$, completing the proof. ■

Bounding Resource Allocations. We model the allocations of request $r \in \mathcal{R}$ on resource $(x, y) \in R_S$ as random variable $A_{r,x,y} \in [0, A_{\text{max}}(r, x, y)]$. We have $\mathbb{P}(A_{r,x,y} = A(m_r^k, x, y)) = f_r^k$ and $\mathbb{P}(A_{r,x,y} = 0) = 1 - \sum_{(f_r^k, m_r^k) \in \mathcal{D}_r} f_r^k$. Let $A_{x,y} = \sum_{r \in \mathcal{R}} A_{r,x,y}$ denote the cumulative allocations induced on resource (x, y) . As $\mathbb{E}(A_{x,y}) = \sum_{r \in \mathcal{R}} \sum_{(f_r^k, m_r^k) \in \mathcal{D}_r} f_r^k \cdot A(m_r^k, x, y)$ holds by definition and using Theorem 18, $\mathbb{E}(A_{x,y}) \leq d_S(x, y)$ is obtained for $(x, y) \in R_S$.

► **Lemma 22.** Consider a node resource $(\tau, u) \in R_S^V$. Choose $0 < \varepsilon \leq 1$, such that $d_{\max}(r, \tau, u)/d_S(\tau, u) \leq \varepsilon$ holds for $r \in \mathcal{R}$. Let $\Delta = \sum_{r \in \mathcal{R}} (A_{\max}(r, \tau, u)/d_{\max}(r, \tau, u))^2$ and $\beta = 1 + \varepsilon \cdot \sqrt{2 \cdot \Delta \cdot \log(|V_S| \cdot |\mathcal{T}|)}$. Then $\mathbb{P}(A_{\tau, u} \geq \beta \cdot d_S(\tau, u)) \leq (|V_S| \cdot |\mathcal{T}|)^{-4}$ holds.

Proof. We apply Hoeffding (cf. Theorem 20) with $t = \varepsilon \cdot \sqrt{2 \cdot \Delta \cdot \log(|V_S| \cdot |\mathcal{T}|)} \cdot d_S(\tau, u)$:

$$\begin{aligned} \mathbb{P}(A_{\tau, u} - \mathbb{E}(A_{\tau, u}) \geq t) &\leq \exp\left(\frac{-4 \cdot \varepsilon^2 \cdot \Delta \cdot \log(|V_S| \cdot |\mathcal{T}|) \cdot d_S^2(\tau, u)}{\sum_{r \in \mathcal{R}} (A_{\max}(r, \tau, u))^2}\right) \\ &\leq \exp\left(\frac{-4 \cdot \varepsilon^2 \cdot \Delta \cdot \log(|V_S| \cdot |\mathcal{T}|) \cdot d_S^2(\tau, u)}{\sum_{r \in \mathcal{R}} (\varepsilon \cdot d_S(\tau, u) \cdot A_{\max}(r, \tau, u)/d_{\max}(r, \tau, u))^2}\right) \\ &= \exp\left(\frac{-4 \log(|V_S| \cdot |\mathcal{T}|) \sum_{r \in \mathcal{R}} (A_{\max}(r, \tau, u)/d_{\max}(r, \tau, u))^2}{\sum_{r \in \mathcal{R}} (A_{\max}(r, \tau, u)/d_{\max}(r, \tau, u))^2}\right) \\ &= (|V_S| \cdot |\mathcal{T}|)^{-4} \end{aligned}$$

Above, we have used $A_{\max}(r, \tau, u) \leq \varepsilon \cdot d_S(\tau, u) \cdot A_{\max}(r, \tau, u)/d_{\max}(r, \tau, u)$, which follows from $d_{\max}(r, \tau, u) \leq \varepsilon \cdot d_S(\tau, u)$. We then plugged in the definition of Δ and reduced the fraction. Lastly, to obtain the lemma's statement, we utilize that the expected allocations $\mathbb{E}(A_{\tau, u})$ are upper bounded by the capacity $d_S(\tau, u)$: $\mathbb{P}(A_{\tau, u} \geq \beta \cdot d_S(\tau, u)) \leq (|V_S| \cdot |\mathcal{T}|)^{-4}$. ■

The probability to violate edge resources can be bounded analogously (see Appendix A for the proof):

► **Lemma 23.** We consider a single edge $(u, v) \in E_S$. Choose $0 < \varepsilon \leq 1$, such that $d_{\max}(r, u, v)/d_S(u, v) \leq \varepsilon$ holds for all $r \in \mathcal{R}$. Let $\Delta = \sum_{r \in \mathcal{R}} (A_{\max}(r, u, v)/d_{\max}(r, u, v))^2$ and $\gamma = 1 + \varepsilon \cdot \sqrt{2 \cdot \Delta \cdot \log |V_S|}$. Then $\mathbb{P}(A_{u, v} \geq \gamma \cdot d_S(u, v)) \leq |V_S|^{-4}$ holds.

Main Result. Given the above, we can now prove that Algorithm 3 indeed is a FPT approximation for the profit variant of the VNEP *with high probability*.

► **Theorem 24.** Assume $|V_S| \geq 3$. Let $0 < \varepsilon \leq 1$ be chosen minimally, such that $d_{\max}(r, x, y)/d_S(x, y) \leq \varepsilon$ holds for $(x, y) \in R_S$ and $r \in \mathcal{R}$. Randomized rounding yields a (α, β, γ) tri-criteria FPT approximation for the profit variant of the VNEP, such that it finds a solution with high probability of at least an $\alpha = 1/3$ fraction of the optimal profit, and cumulative allocations within factors of β (nodes) and γ (edges) of the original capacities, for $\beta = 1 + \varepsilon \cdot \sqrt{2 \cdot \Delta(R_S^V) \cdot \log(|V_S| \cdot |\mathcal{T}|)}$ and $\gamma = 1 + \varepsilon \cdot \sqrt{2 \cdot \Delta(E_S) \cdot \log |V_S|}$ with $\Delta : \mathcal{P}(R_S) \rightarrow \mathbb{R}_{\geq 0}$ being defined as $\Delta(X) = \max_{(x, y) \in X} \sum_{r \in \mathcal{R}} \left(\frac{A_{\max}(r, x, y)}{d_{\max}(r, x, y)}\right)^2$.

Proof. We consider the probability of the rounding step failing to produce a (α, β, γ) -approximate solution. By applying a union bound for any node and edge resource exceeding β or γ times the capacity (cf. Lemma 22 and Lemma 23) together with the probability of *not* achieving 1/3 of the LP's objective (cf. Lemma 21), the probability to not find a solution is upper bounded by 19/20. Hence, the probability of finding an approximate solution within N iterations is at least $1 - (19/20)^N$, i.e., Algorithm 3 will produce such a solution *with high probability*. With respect to the runtime of Algorithm 3 we note that the LP Formulation 3 can be solved in $\mathcal{O}\left(\text{poly}(\sum_{r \in \mathcal{R}} |G_r| \cdot |G_S|^{\text{ew}_X(G_r^X)})\right)$ by e.g. using the Ellipsoid algorithm [15] (cf. 17) and the decomposition algorithm's runtime has the same runtime. Hence, the runtime of Algorithm 3 is bounded by $\mathcal{O}\left(N \cdot \text{poly}(\sum_{r \in \mathcal{R}} |G_r| \cdot |G_S|^{\text{ew}_X(G_r^X)})\right)$, when performing at most N rounding tries. Hence Algorithm 3 is fixed-parameter tractable with respect to the maximal width of any of the extraction orders. ■

Algorithm 4: Randomized Rounding Algorithm for the VNEP (Cost)

```

1 compute LP solution to Formulation 3 for request set  $\mathcal{R}$  subject to
   the objective  $\min \sum_{(x,y) \in R_S, r \in \mathcal{R}} c_S(x,y) \cdot a_r^{x,y}$  and
   the additional constraints  $x_r = 1$  for all  $r \in \mathcal{R}$ 
2 decompose LP solution into convex combinations  $\mathcal{D}_r$  for all  $r \in \mathcal{R}$ 
3 foreach  $r \in \mathcal{R}$  do // postprocess decomposition: prune costly mappings
4   let  $WC_r = \sum_{(f_r^k, m_r^k) \in \mathcal{D}_r} f_r^k \cdot c(m_r^k)$ 
5   remove tuples  $(f_r^k, m_r^k)$  from  $\mathcal{D}_r$  with  $c(m_r^k) > 2 \cdot WC_r$ 
6   normalize weights of  $\mathcal{D}_r$ , such that  $\sum_{(f_r^k, m_r^k) \in \mathcal{D}_r} f_r^k = 1$  holds again
7 do // perform randomized rounding
8   construct solution by choosing mapping  $m_r^k$  with probability  $f_r^k$  for all  $r \in \mathcal{R}$ 
9 while the solution is not  $(\alpha, \beta, \gamma)$ -approximate and maximal rounding tries are not exceeded;

```

4.2 Approximating the Cost Variant

Similarly to the approximation of the profit, the cost variant can be approximated as presented in Algorithm 4. In particular, two changes are necessary compared to Algorithm 3: the preprocessing in Lines 1-3 can be dropped and a postprocessing of the found decompositions needs to be added. Concretely, to obtain a constant approximation of the cost, the decision which of the mappings m_r^k to choose for request $r \in \mathcal{R}$ cannot be purely left to chance. Denoting the cost of a mapping $m_r^k \in \mathcal{M}_r$ by $c(m_r^k) = \sum_{(x,y) \in R_S} c_S(x,y) \cdot A(m_r^k, x, y)$, the set of convex combinations $\mathcal{D}_r = \{(f_r^k, m_r^k)\}_k$ may contain mappings m_r^k of arbitrarily small weight f_r^k while having an arbitrarily high cost $c(m_r^k)$. Hence, if the possibility exists to choose such a mapping in the rounding step, no bound on the rounded cost can be given in general. Hence, the key idea is to remove all fractional mappings of high cost while not losing too much weight in the convex combination. Concretely, given a request $r \in \mathcal{R}$, we denote by $WC_r = \sum_{(f_r^k, m_r^k) \in \mathcal{D}_r} f_r^k \cdot c(m_r^k)$ the *weighted (averaged) cost* of request $r \in \mathcal{R}$ and the algorithm removes all mappings m_r^k from the convex combinations for which $c(m_r^k) > 2 \cdot WC_r$ hold. As the following lemma shows, the sum of the associated weights of the mappings removed must be less than $1/2$:

► **Lemma 25.** *The sum of the weights f_r^k of the mappings m_r^k with cost smaller than two times WC_r is at least $1/2$ for each request $r \in \mathcal{R}$.*

Proof. Let $\lambda_r = \sum_{(f_r^k, m_r^k) \in \mathcal{D}_r: c(m_r^k) \leq 2 \cdot WC_r} f_r^k$ denote the sum of the weights of the mappings of cost bounded by $2 \cdot WC_r$. For the sake of contradiction, assume that $\lambda_r < 1/2$ holds for any request $r \in \mathcal{R}$. By the definition of WC_r and the assumption on λ_r , we obtain the following contradiction:

$$WC_r = \sum_{(f_r^k, m_r^k) \in \mathcal{D}_r} f_r^k \cdot c(m_r^k) \tag{23}$$

$$\leq \sum_{(f_r^k, m_r^k) \in \mathcal{D}_r: c(m_r^k) > 2 \cdot WC_r} f_r^k \cdot c(m_r^k) \tag{24}$$

$$\leq \sum_{(f_r^k, m_r^k) \in \mathcal{D}_r: c(m_r^k) > 2 \cdot WC_r} f_r^k \cdot 2 \cdot WC_r \tag{25}$$

$$\leq (1 - \lambda_r) \cdot 2 \cdot WC_r < WC_r \tag{26}$$

The validity of Equation 25 follows as all the considered decompositions have a cost of at

least two times WC_r and the first inequality of Equation 26 follows as $(1 - \lambda_r) > 1/2$ holds by assumption. Lastly, Equation 26 yields the contradiction, showing that indeed $\lambda_r \geq 1/2$ holds for $r \in \mathcal{R}$. \blacksquare

Deterministic Guarantee for the Cost. It is easy to establish that initially, i.e., before pruning mappings from \mathcal{D}_r , the cost of an optimal solution equals the sum of the (fractional) costs of the convex combinations:

► **Lemma 26.** *Letting C_{LP} denote the cost of the LP solution computed in Line 1, we have:*

$$\sum_{r \in \mathcal{R}} WC_r = \sum_{r \in \mathcal{R}, D_r^k \in \mathcal{D}_r} f_r^k \cdot c(m_r^k) = C_{LP}. \quad (27)$$

Proof. Due to the definition of the objective, we have $C_{LP} = \sum_{(x,y) \in R_S, r \in \mathcal{R}} c_S(x,y) \cdot a_r^{x,y}$. From the correctness of the decomposition algorithm (cf. Theorem 18) we know that $a_r^{x,y} \geq \sum_k f_r^k \cdot A(m_r^k, x, y)$ holds for each request $r \in \mathcal{R}$ and each resource $(x, y) \in R_S$. As the cost $c(m_r^k)$ of a mapping m_r^k equals $\sum_{(x,y) \in R_S} c_S(x,y) \cdot A(m_r^k, x, y)$, the equality $\sum_{r \in \mathcal{R}, D_r^k \in \mathcal{D}_r} f_r^k \cdot c(m_r^k) = C_{LP}$ follows. Lastly, as the equality $\sum_{r \in \mathcal{R}} WC_r = \sum_{r \in \mathcal{R}, (f_r^k, m_r^k) \in \mathcal{D}_r} f_r^k \cdot c(m_r^k)$ holds by definition of WC_r . \blacksquare

► **Lemma 27.** *The cost of any solution returned by the randomized rounding scheme is upper bounded by two times the optimal cost.*

Proof. Let C_{LP} denote the cost of the optimal solution returned by LP Formulation 3. By allowing to select only decompositions m_r^k for which $c(m_r^k) \leq 2 \cdot WC_r$ holds and denoting the selected mapping by \hat{m} we have $c(\hat{m}_r) \leq 2 \cdot WC_r$. Hence $\sum_{r \in \mathcal{R}} c(\hat{m}_r) \leq 2 \cdot \sum_{r \in \mathcal{R}} WC_r$ holds. Together with Lemma 26 we obtain $\sum_{r \in \mathcal{R}} c(\hat{m}_r) \leq 2 \cdot C_{LP}$. As C_{LP} is a lower bound for the minimum cost C_{opt} of the optimal solution the lemma holds. \blacksquare

Bounding Resource Allocations. The employed rounding scheme for the cost variant of the VNEP equals the one for the profit. Revisiting the analysis of the probabilistic guarantees on the capacity violations for the profit approximation, we note that the analysis only depended on the fact that the expected load on any resource is upper bounded by its capacity. Due to the rescaling, this does not hold anymore. However, as the weights are scaled by at most a factor of 2 (cf. Lemma 25), it is easy to establish the following:

► **Lemma 28.** $\mathbb{E}(A_{x,y}) \leq 2 \cdot d_S(x, y)$ holds for all resources $(x, y) \in R_S$.

Plugging in this doubled expected load into the proofs of Lemmas 22 and 23, the approximation factors become $\beta = 2 + \varepsilon \cdot \sqrt{2 \cdot \Delta \cdot \log(|V_S| \cdot |\mathcal{T}|)}$ and $\gamma = 2 + \varepsilon \cdot \sqrt{2 \cdot \Delta \cdot \log(|V_S|)}$, respectively, and we state the following lemmas without proof.

► **Lemma 29.** *Consider a node resource $(\tau, u) \in R_S^V$. Choose $0 < \varepsilon \leq 1$, such that $d_{\max}(r, \tau, u)/d_S(\tau, u) \leq \varepsilon$ holds for $r \in \mathcal{R}$. Let $\Delta = \sum_{r \in \mathcal{R}} (A_{\max}(r, \tau, u)/d_{\max}(r, \tau, u))^2$ and $\beta = 2 + \varepsilon \cdot \sqrt{2 \cdot \Delta \cdot \log(|V_S| \cdot |\mathcal{T}|)}$. Then $\mathbb{P}(A_{\tau,u} \geq \beta \cdot d_S(\tau, u)) \leq (|V_S| \cdot |\mathcal{T}|)^{-4}$ holds, when approximating the cost variant of the VNEP.*

► **Lemma 30.** *We consider a single edge $(u, v) \in E_S$. Choose $0 < \varepsilon \leq 1$, such that $d_{\max}(r, u, v)/d_S(u, v) \leq \varepsilon$ holds for all $r \in \mathcal{R}$. Let $\Delta = \sum_{r \in \mathcal{R}} (A_{\max}(r, u, v)/d_{\max}(r, u, v))^2$ and $\gamma = 2 + \varepsilon \cdot \sqrt{2 \cdot \Delta \cdot \log |V_S|}$. Then $\mathbb{P}(A_{u,v} \geq \gamma \cdot d_S(u, v)) \leq |V_S|^{-4}$ holds, when approximating the cost variant of the VNEP.*

Main Result. Given the Lemmas 27, 29, and 30, a result analogous to Theorem 24 can be obtained for the cost variant of the VNEP.

► **Theorem 31.** *Assume that $|V_S| \geq 3$. Let $0 < \varepsilon \leq 1$ be chosen minimally, such that $d_{\max}(r, x, y)/d_S(x, y) \leq \varepsilon$ holds for $(x, y) \in R_S$ and $r \in \mathcal{R}$. Randomized rounding yields a (α, β, γ) tri-criteria FPT approximation for the cost variant of the VNEP, such that it finds a allocation with high probability of cost at most $\alpha = 2$ times the optimal cost, and cumulative allocations within factors of β (nodes) and γ (edges) of the original capacities, for $\beta = 2 + \varepsilon \cdot \sqrt{2 \cdot \Delta(R_S^V) \cdot \log(|V_S| \cdot |\mathcal{T}|)}$ and $\gamma = 2 + \varepsilon \cdot \sqrt{2 \cdot \Delta(E_S) \cdot \log |V_S|}$ with $\Delta : \mathcal{P}(R_S) \rightarrow \mathbb{R}_{\geq 0}$ being defined as $\Delta(X) = \max_{(x,y) \in X} \sum_{r \in \mathcal{R}} \left(\frac{A_{\max}(r, x, y)}{d_{\max}(r, x, y)} \right)^2$.*

5 Extraction Width: Graph Classes and Complexity

The runtime of our approximation algorithms grows exponentially in the maximal width of any extraction order. Hence, three questions arise: (i) are there any polynomial-time (unparameterized) approximations for the VNEP, (ii) which graph classes have a bounded extraction width, and (iii) how can extraction orders of minimal width be computed?

The first question can be answered quite easily: there cannot exist polynomial-time approximations (unless $\mathcal{P} = \mathcal{NP}$), as the following theorem shows.

► **Theorem 32.** *The fractional VNEP is \mathcal{NP} -hard and inapproximable (unless $\mathcal{P} = \mathcal{NP}$). This even holds when request graphs are planar.*

Proof. The authors of this paper have recently shown in [27] that finding *valid* mappings for planar requests is \mathcal{NP} -complete when allowing for node and edge placement restrictions. Hence, optimizing over a convex combination of *valid mappings* is \mathcal{NP} -hard and the fractional VNEP is (polynomial-time) inapproximable (unless $\mathcal{P} = \mathcal{NP}$). This even holds when not considering substrate graph capacities. ■

The above theorem validates our FPT approach to computing solutions to the fractional VNEP, as there cannot exist polynomial-time algorithms for general request graphs and unless $\mathcal{P} = \mathcal{NP}$ holds. It furthermore underlines that the complexity of the request graphs must be reflected when computing solutions to the fractional VNEP.

5.1 Graph Classes of Bounded Extraction Width

Given the impossibility of polynomial-time approximations for arbitrary request graphs, we now study graph classes that have bounded extraction width. In particular, we show that ‘cactus graph requests’ and generalizations thereof have bounded extraction width.

► **Theorem 33.** *Consider a cactus request graph G_r , i.e., one for which cycles in its undirected interpretation intersect in at most a single node. Then $\text{ew}_{\mathcal{X}}(G_r^{\mathcal{X}}) \leq 2$ holds for any extraction order $G_r^{\mathcal{X}}$. Hence, our approximations run in polynomial-time for cactus graph requests.*

Proof. Consider any extraction order $G_r^{\mathcal{X}}$. $|\mathcal{L}_{r,e}^{\mathcal{X}}| \leq 1$ must hold for all edges $e \in E_r^{\mathcal{X}}$: if this was not the case then two confluences would overlap in e and violate the cactus property. Thus, edge label sets are either equal or disjoint and the maximal edge bag size is 1. ■

While the class of cactus graphs is restrictive, it can be shown that by adding edges parallel to existing ones the width increases by at most the maximum degree of the graph:

► **Lemma 34.** *Given an arbitrary graph G_r , adding any number of parallel edges (of any direction) for an existing edge does increase the extraction width of G_r by at most the maximum degree of G_r . This also holds true if instead paths are added instead of edges.*

Proof. Consider an extraction order G_r^χ minimizing the width. Let $e = (i, j) \in E_r$ be an existing edge and assume without loss of generality that the orientation of the edge e is the same in G_r^χ . Now, when adding another edge $e' = (i, j)$ or $e'' = (j, i)$ to E_r , we orient the edge the same way as the original edge e . Hence, e' would be introduced to E_r^χ as is, and the orientation of e'' would be reversed. Now, if the node j was previously not the target of a confluence, then by introducing e' or e'' a new confluence was created and the size of the edge bag of node i containing the edge $(i, j) \in E_r^\chi$ increases by one. However, adding e' or e'' cannot introduce confluences beyond that: the addition of a *parallel* edge cannot enable a novel confluence to be created. Hence, arbitrarily many parallel edges can be added while increasing the size of an edge bag of the tail of the edge by at most one per outgoing edge. Hence, arbitrarily many parallel edges can be created for any existing edge while increasing the sizes of a single edge bag per node by at most one per outgoing edge. Hence, the extraction width of the resulting graph has increased by at most the maximum degree of the original graph G_r . ■

Lastly, we note that the examples depicted in Figure 1 have small extraction widths.

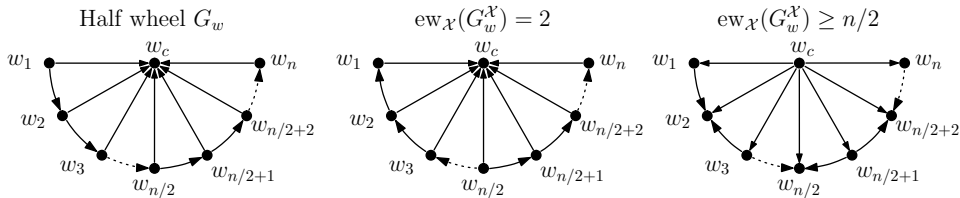
► **Observation 35.** *The example request graphs depicted in Figure 1 have an extraction width of 2 and 3 respectively.*

Proof. The request graph depicted on the right of Figure 1 is a cactus graph and hence has width 2 by Lemma 33.

Considering the request graph on the left, we note that when only considering the solid edges, the graph is a cactus and the width is hence 2. By adding the ‘parallel’ dashed edges, the extraction width increases by at most the maximum degree 3 of the cactus graph according to Lemma 34. However, considering the proof of Lemma 34, we see that for the node LB_1 , only 2 outgoing edges exist (according to the solid edges) and hence the width increases by at most 2. Furthermore, as the node LB_2 is already the target of a confluence, the overall extraction width increases by 1 and the width is hence 3. ■

5.2 Hardness of Computing Extraction Orders

Given the above examples, we now study the computational complexity of finding extraction orders of minimum width. In fact, we prove the \mathcal{NP} -hardness of computing optimal extraction orders. Our prove relies on a reduction from vertex cover. As a first step towards this goal, consider the following lemma.



■ **Figure 6** Depicted is an arbitrarily oriented ‘half wheel’ (left) together with two extraction orders: The extraction order in the center is rooted at $w_{n/2}$ and has width 2. The other order on the right is rooted at w_c and has a width of at least $n/2$ (shown below).

► **Lemma 36.** *Consider the half wheel graph G_w depicted in Figure 6 and any extraction order $G_w^{\mathcal{X}}$ being rooted at w_c . Letting $\text{VC} = \{w_k \in V_w \mid (w_{k-1}, w_k) \in E_r^{\mathcal{X}} \vee (w_{k+1}, w_k) \in E_r^{\mathcal{X}}\}$, the following holds: $\text{ew}_{\mathcal{X}}(G_w^{\mathcal{X}}) = |\text{VC}| + 1$.*

Proof. We denote by $G_w^i = (V_w^i, E_W^i)$ the subgraph of $G_w^{\mathcal{X}}$ induced by the set of nodes $\{w_1, \dots, w_i\} \cup \{w_c\}$. Let $\text{VC}_i = \{w_k \in V_w^i \mid (w_{k-1}, w_k) \in E_r^{\mathcal{X}} \vee (w_{k+1}, w_k) \in E_r^{\mathcal{X}}\}$.

Via induction over the subgraphs G_w^i it can be seen that the edges $e_k = (w_c, w_k)$, and $e_{k-1} = (w_c, w_{k-1})$ are either both labeled by w_k (if $w_k \in \text{VC}_i$) or by $w_{k-1} \in \text{VC}_i$ (if $w_k \notin \text{VC}_i$) for all $k \in \{2, \dots, i\}$.

Observing that $\text{VC}_n = \text{VC}$ equals the labels introduced in $G_w^{\mathcal{X}}$ and noting that the edge label sets $\mathcal{L}_{r, e_i}^{\mathcal{X}}$ and $\mathcal{L}_{r, e_{i+1}}^{\mathcal{X}}$ overlap for all $i \in \{1, \dots, n-1\}$, the root w_c must have a single edge bag containing all the labels contained in $|\text{VC}_i|$. Hence, $\text{ew}_{\mathcal{X}}(G_w^{\mathcal{X}}) \geq |\text{VC}| + 1$ follows. Furthermore, only the nodes contained in $|\text{VC}|$ can be labels (a node not contained in VC has only a single incoming edge) and the result follows. ■

By Lemma 36, the following corollary is immediate.

► **Corollary 37.** *Consider a wheel graph G_w with n outer nodes. Considering any extraction order $G_w^{\mathcal{X}}$ for which the node w_c is chosen to be the root, $\text{ew}_{\mathcal{X}}(G_w^{\mathcal{X}}) \geq \lfloor n/2 \rfloor + 1$ holds.*

The result of Lemma 36 can be generalized in the following sense.

► **Lemma 38.** *Given is a connected, undirected graph $\bar{G} = (\bar{V}, \bar{E})$, we define a directed version $G_{VC} = (V_{VC}, E_{VC})$ with an additional super node \hat{r} as follows: $V_{VC} = \bar{V} \sqcup \{\hat{r}\}$, and $E_{VC} = \{(i, j) \mid \{i, j\} \in \bar{E}, i < j\} \cup \{(\hat{r}, i) \mid i \in \bar{V}\}$. The minimal width of an extraction order $G_{VC}^{\mathcal{X}}$ rooted at \hat{r} equals the size of the minimum vertex cover of \bar{G} plus one.*

Proof. Let $G_{VC}^{\mathcal{X}}$ be an extraction order of G_{VC} which is rooted at \hat{r} . The proof of Lemma 36 has shown that whenever a path P in the original graph is considered, all nodes of $G_{VC}^{\mathcal{X}}$ with at least one incoming edge (with respect to the original edge set) are labels of the *same* edge bag of the root \hat{r} . As this property holds for any simple path contained in \bar{G} and as \bar{G} is connected, there can only be a single edge bag; if there was more than one edge bag, then there does not exist a path P connecting any of the edges of the first bag to any of the edges in the second bag, refuting the connectivity of \bar{G} . Applying Lemma 36 for any path P of the original graph, the single edge bag of the root \hat{r} must contain any node having at least one incoming edge according to the original edge set \bar{E} . Hence, assuming that $G_{VC}^{\mathcal{X}}$ has minimal width, the width of $G_{VC}^{\mathcal{X}}$ equals the size of the minimum vertex cover of \bar{G} plus one. ■

Lemma 38 is the basis of our proof that computing extraction orders of minimal width is \mathcal{NP} -hard via a reduction from vertex cover (cf. Theorem 40). For the proof of our reduction, we require the following lemma.

► **Lemma 39.** *Consider a graph $G = (V, E)$ with a corresponding extraction order $G^{\mathcal{X}} = (V, E^{\mathcal{X}}, s)$. Assume that a node $v \in V$ exists that separates a set of nodes $U \subset V$ from the root node s . Then any edge incident to v and some node $u \in U$ is oriented away from v in the extraction order $G^{\mathcal{X}}$, i.e. $(v, u) \in E^{\mathcal{X}}$ holds for all $u \in U$.*

Proof. Assume for the sake of contradiction that for some node $u \in U$ the edge (u, v) is contained in the extraction order. By the definition of the extraction order all nodes must be reachable from the root s . As the node v separates u from the root, all paths from s to u must contain v . Hence, u must be reachable from v and the edge (u, v) hence creates a loop in $G^{\mathcal{X}}$ which contradicts the acyclicity of $G^{\mathcal{X}}$. Hence, any edge incident to v and some node $u \in U$ must be directed away from v . ■

► **Theorem 40.** *Computing an extraction order of minimum width is \mathcal{NP} -hard.*

Proof. We give a polynomial time reduction of the vertex cover problem to the problem of finding the extraction order of minimum width. We adapt the construction used in Lemma 38 slightly, to force the mapping of the root node to \hat{r} . Concretely, we add a *half wheel graph* (cf. Figure 6) G_w with $2 \cdot |\bar{V}| + 2$ outer nodes to the graph G_{VC} and identify the node \hat{r} with the wheel's node w_c , i.e. $\hat{r} = w_c$. Let $G_{VC}^{\mathcal{X}} = (V_{VC}, E_{VC}^{\mathcal{X}}, s_{VC})$ be an extraction order of minimum width. The extraction order's root s_{VC} must be placed on some outer wheel node:

Root s_{VC} is placed on a wheel node w_i : We first consider the orientations of edges inside the wheel graph. According to Figure 6 (center) there is an orientation such that the extraction width inside the wheel graph is 2. The node $\hat{r} = w_c$ separates the original graph \bar{G} from the extraction order's root $s_{VC} = w_i$. Thus, by Lemma 39 all edges incident to a node $v \in \bar{V}$ and w_c must be oriented away from w_c . Hence, excluding the outer wheel nodes, the node w_c is a root in the corresponding extraction order. Thus, the width of the extraction order $G_{VC}^{\mathcal{X}}$ – excluding the outer wheel nodes – equals the size of a minimum vertex cover of \bar{G} plus one by Lemma 38 and the assumption that $G_{VC}^{\mathcal{X}}$ is of minimal width. Lastly, note that no confluence spanning the wheel graph G_w and the graph \bar{G} exists. Letting VC denote a minimal vertex cover of \bar{G} , the width of the extraction order $G_{VC}^{\mathcal{X}}$ equals $\max\{2, |VC|\} \leq |\bar{V}|$.

Root s_{VC} is placed on $\hat{r} = w_c$: In this case, the width of the extraction order $G_{VC}^{\mathcal{X}}$ is at least $|\bar{V}| + 1$ based on Corollary 37. Hence, as the size of a vertex cover of \bar{G} is always less than $|\bar{V}|$, the placement of the root on \hat{r} contradicts the optimality assumption of $G_{VC}^{\mathcal{X}}$.

Root s_{VC} is placed on a node $v \in \bar{V}$: In this case, the width is again at least $|\bar{V}| + 1$: as the node $\hat{r} = w_c$ separates the outer wheel nodes from the root s_{VC} , all wheel edges incident to w_c are oriented away from w_c . As w_c is hence a root in the extraction order restricted to the wheel graph, the width is at least $|\bar{V}| + 1$ by Lemma 36. Thus, the placement of the extraction's root on any node $v \in \bar{V}$ contradicts the optimality of the extraction order $G_{VC}^{\mathcal{X}}$.

Now, let $VC \subseteq \bar{V}$ denote a minimum vertex cover of \bar{G} . If $|VC| > 1$ holds, then for any optimal extraction order $G_{VC}^{\mathcal{X}}$, $\text{ew}_{\mathcal{X}}(G_{VC}^{\mathcal{X}}) = |VC| + 1$ holds. Furthermore, a minimal vertex cover VC can be recovered from any minimum width extraction order $G_{VC}^{\mathcal{X}}$ by placing any node v in the cover VC whenever at least two edges are oriented towards it in $G_{VC}^{\mathcal{X}}$. As the cases in which the minimal vertex cover is less or equal to 1 can be trivially identified, computing a minimum width extraction order is \mathcal{NP} -hard. ■

6 Conclusion

We have presented the first (fixed-parameter tractable) approximation algorithms for the Virtual Network Embedding Problem (VNEP) supporting arbitrary request graphs. To enable the decomposability of general request graphs, we have developed a novel LP formulation whose size is parameterized by a novel graph number: the extraction width and exploring it further will be of great interest for practical applications. Finally, while having focused on the theoretic aspects of approximating the VNEP, we provide the research community with implementations and empirical evaluations at <https://vnep-approx.github.io/>.

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A Deferred Proofs

► **Lemma 23.** *We consider a single edge $(u, v) \in E_S$. Choose $0 < \varepsilon \leq 1$, such that $d_{\max}(r, u, v)/d_S(u, v) \leq \varepsilon$ holds for all $r \in \mathcal{R}$. Let $\Delta = \sum_{r \in \mathcal{R}} (A_{\max}(r, u, v)/d_{\max}(r, u, v))^2$ and $\gamma = 1 + \varepsilon \cdot \sqrt{2 \cdot \Delta \cdot \log |V_S|}$. Then $\mathbb{P}(A_{u,v} \geq \gamma \cdot d_S(u, v)) \leq |V_S|^{-4}$ holds.*

Proof. Each random variable $A_{r,u,v}$ is clearly contained in the interval $[0, A_{\max}(r, u, v)]$. We choose $t = \varepsilon \cdot \sqrt{2 \cdot \Delta \cdot \log |V_S|} \cdot d_S(u, v)$ and apply Hoeffding’s Inequality:

$$\begin{aligned} & \mathbb{P}(A_{u,v} - \mathbb{E}(A_{u,v}) \geq t) \\ & \leq \exp\left(\frac{-2 \cdot t^2}{\sum_{r \in \mathcal{R}} (A_{\max}(r, u, v))^2}\right) \end{aligned} \quad (28)$$

$$\leq \exp\left(\frac{-4 \cdot \varepsilon^2 \cdot \Delta \cdot \log |V_S| \cdot d_S^2(u, v)}{\sum_{r \in \mathcal{R}} (\varepsilon \cdot d_S(\tau, u) \cdot A_{\max}(r, u, v)/d_{\max}(r, u, v))^2}\right) \quad (29)$$

$$\leq \exp\left(\frac{-4 \cdot \varepsilon^2 \cdot \sum_{r \in \mathcal{R}} (A_{\max}(r, u, v)/d_{\max}(r, u, v))^2 \cdot \log |V_S| \cdot d_S^2(u, v)}{\varepsilon^2 \cdot d_S^2(u, v) \cdot \sum_{r \in \mathcal{R}} (A_{\max}(r, u, v)/d_{\max}(r, u, v))^2}\right) = |V_S|^{-4} \quad (30)$$

We have used $A_{\max}(r, \tau, u) \leq \varepsilon \cdot d_S(\tau, u) \cdot A_{\max}(r, \tau, u)/d_{\max}(r, \tau, u)$ in the second step, which follows from $d_{\max}(r, \tau, u) \leq \varepsilon \cdot d_S(\tau, u)$. In the third step we plugged in the definition of Δ . Lastly, we utilize that the expected load $\mathbb{E}(A_{\tau,u})$ is upper bounded by the resource’s capacity $d_S(\tau, u)$ to obtain the lemma’s statement. ■