On the Time Complexity of Distributed Topological Self-Stabilization

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Joint work with Dominik Gall, Riko Jacob, Christian Scheideler, Stefan Schmid, and Hanjo Täubig

Arizona State University

LATIN 2010

Reminder: Purpose of a Model

A model should reflect reality

- accurately enough
- simply enough

to say something interesting.

The execution time of a parallel/distributed algorithm heavily depends on how parallelism is modeled:

- too loose: too many operations (some possibly conflicting) executed in one round
- too rigid: may “force” a sequential execution of the operations when parallelism could still be exploited

We present an execution framework for local topological algorithms, which sheds new light on the *achievable paralellism*

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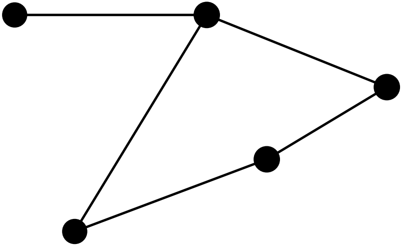
We present an execution framework for local topological algorithms, which sheds new light on the *achievable paralellism*

Topological (Self-Stabilizing) Algorithm

System State
Graph G on n nodes

Local actions
change edges between neighbors

Goal
Achieve a specific target topology



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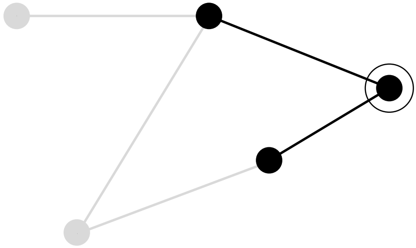

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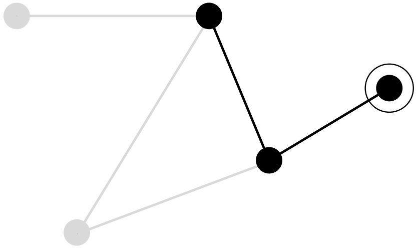

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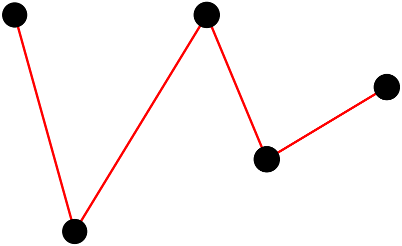

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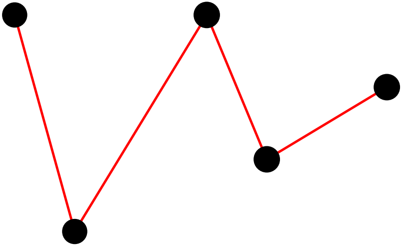


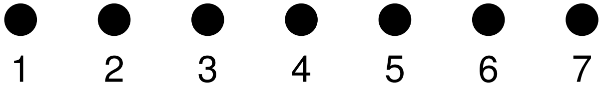
Outline

Linearization

Problem

Initial State Arbitrary connected graph on $\{1, \ldots, n\}$

Goal Edges of the form $\{i, i+1\}$



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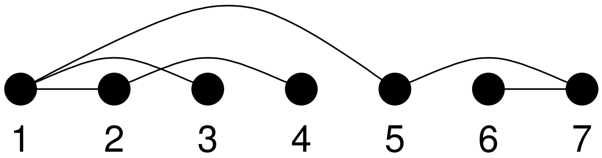

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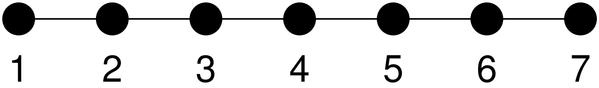

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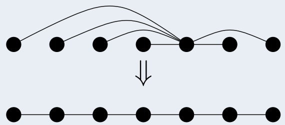
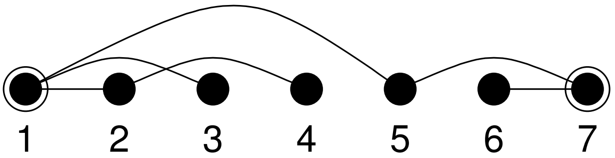


$t = 0$



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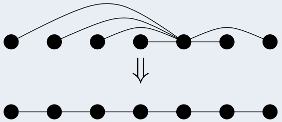
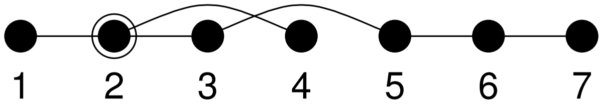


$t = 1$



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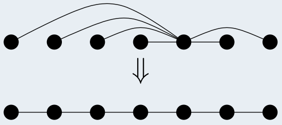

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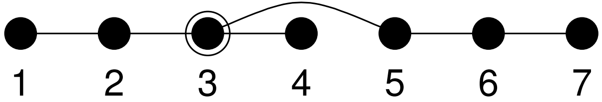

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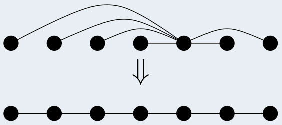

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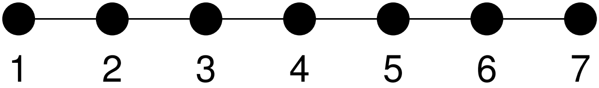

Known result

Theorem [Onus, Richa, Scheideler '06]

The presented linearization algorithm takes $\Theta(n)$ rounds

Lower bound

Upper bound

For every missing edge $\{k, k+1\}$:
Consider the length of the shortest interval $[i, j]$ s.t. k and $k+1$ are connected in the subgraph induced by $\{i, \ldots, k, \ldots, j\}$.
This length is reduced in every step by at least one.

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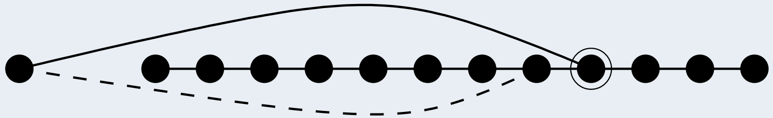

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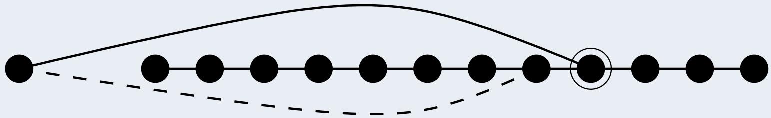

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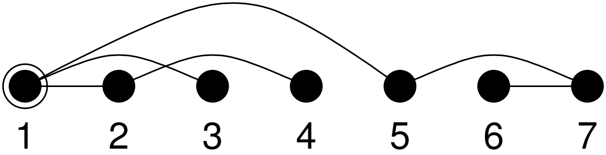
Criticism

High degree nodes

Executing the rule should take time proportional to the degree.

Concurrent access

Nodes should not participate “passively” in more than one execution of the rule.



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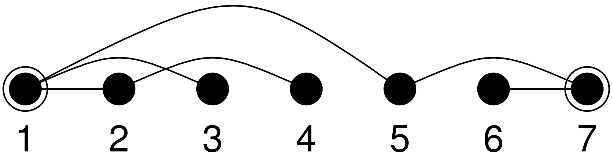
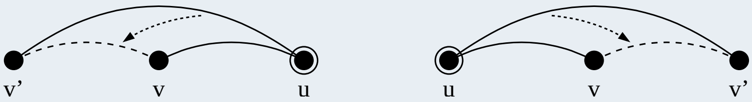


Atomic Rule

Atomic algorithmic rule



Every such triple is considered independently

The previous algorithm: Example with 3 triples

Idea: Scheduler

In every round, a scheduler decides on the executed triples.
For example, an independent set of rules (matching)

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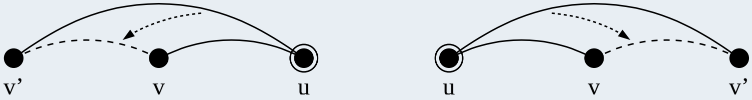

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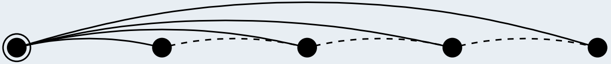

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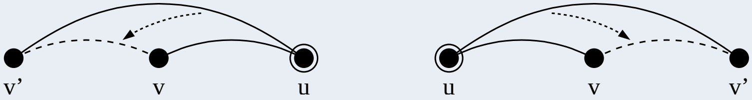

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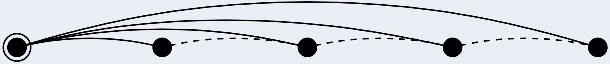

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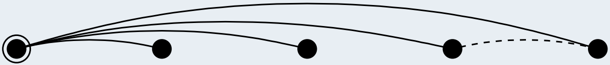
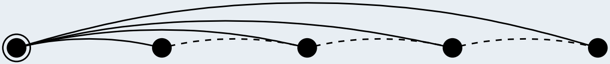
Variants of the Algorithm

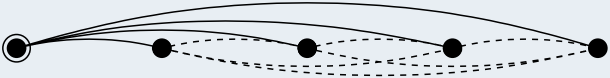
Only longest: LinMax



All neighbors: LinNgb



All possible triples: LinAll



Framework for different models of execution

Rounds

Execution progresses in synchronous rounds;
these are counted

Scheduler

In every round, the set of active rules is determined, and a scheduler decides which rules are executed in this round

Matching

The rules define hyperedges on 3 nodes.
A matching is a set of hyperedges that do not share nodes.

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Schedulers: What happens in one round?

Best Case Scheduler

Algorithm chooses a matching

Realistic parallelism, but centrally coordinated

Randomized Scheduler

Takes random maximal matching

Worst Case Scheduler

An adversary chooses dominating set of actions

Very realistic, even if not synchronized

All rules [Onus, Richa, Scheideler '06]

Fire all allowed actions; insertion is stronger than deletion

Unrealistic, easy to analyze.

Antichain [Critical Path, Blumofe, Leiserson '93]

For some fixed serial execution, the next antichain

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Algorithm LinAll with best case scheduler

Theorem

There exists a scheduler such that for any connected initial graph G_0 the algorithm LinAll converges in $O(n \log n)$ steps.

Proof

Define potential $\sum_{(i,j)\in E} |j - i|$
Scheduler: choose matching greedily according to potential ("longest and second longest edge on highest degree node").
One matching reduces potential by factor $1 - 1/\Theta(n)$.
Round k can happen if $n^3(1 - 1/cn)^k \geq n - 1$; solve for k

Summary of Linearization Results

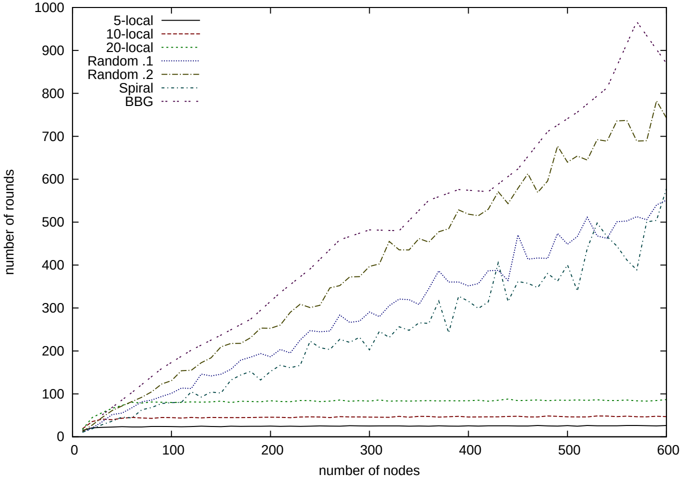
Summary

algorithm	scheduler	time	work
*	*	$\Omega(n)$	$\Omega(n)$
LinNgb	all	$O(n)$	$O(n^2)$
LinAll	best-case	$O(n \log n)$	$O(n^2 \log n)$
LinAll	worst-case	$O(n^2 \log n)$	$O(n^3)$
LinAll	critical-path	$\Theta(n^3)$	$\Theta(n^3)$
LinMax	worst-case	$\Theta(n^2)$	$\Theta(n^2)$
LinMax	critical-path	$O(n^2)$	$O(n^2)$

Open Questions

LinMax or LinAll with best-case or randomized scheduler $\Theta(n)$?

Experimental Results: LinAll, Randomized Scheduler



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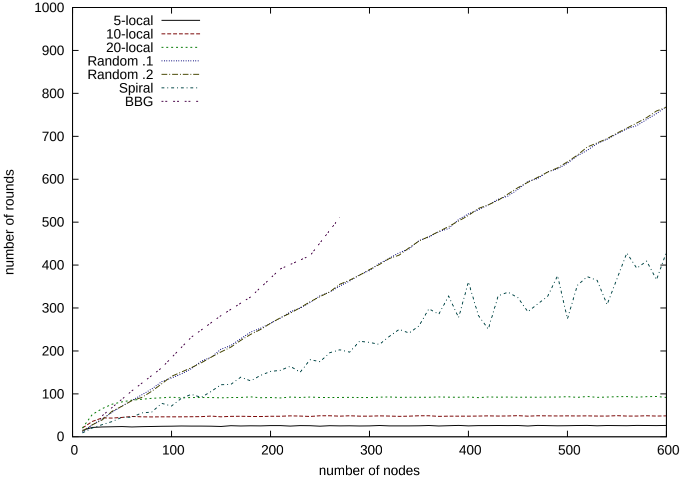

More future work

- Can we devise local, distributed algorithms that implement (approximations of) the proposed schedulers?

Thank you!
Questions?