# Competitive and Fair Throughput for Co-Existing Networks Under Adversarial Interference

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## ABSTRACT

This paper initiates the formal study of a fundamental problem: How to efficiently allocate a shared communication medium among a set of $K$ co-existing networks in the presence of arbitrary external interference? While most literature on medium access focuses on how to share a medium among *nodes*, these approaches are often either not directly applicable to co-existing networks as they would violate the independence requirement, or they yield a low throughput if applied to multiple networks. We present the randomized medium access (MAC) protocol COMAC which guarantees that a given communication channel is shared fairly among competing and independent networks, and that the available bandwidth is used efficiently. These performance guarantees hold in the presence of arbitrary external interference or even under adversarial jamming. Concretely, we show that the co-existing networks can use a $\Omega(\varepsilon^2 \min\{\varepsilon, 1/poly(K)\})$-fraction of the non-jammed time steps for successful message transmissions, where $\varepsilon$ is the (arbitrarily distributed) fraction of time which is not jammed.

## Categories and Subject Descriptors

C.2.5 [**Computer-Communication Networks**]: Local and Wide-Area Networks—*Access schemes*; F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems—*Sequencing and scheduling*

## General Terms

Algorithms, Reliability, Theory

## Keywords

Wireless Ad-hoc Networks, MAC Protocols, Jamming

## 1. INTRODUCTION

The decentralized allocation of a communication medium among a set of wireless nodes does not only constitute one of the most fundamental theoretical problems in distributed computing, but is also of direct practical relevance. Today, a chunk of the wireless spectrum is often simultaneously used by many devices belonging to different, so-called *co-existing networks*. It is expected that the popularity of wireless mobile devices will further increase the resource sharing by such networks in the future.

Interestingly, not much is known today on how a given spectrum can be shared efficiently and fairly among co-existing networks, especially in environments with uncontrollable external interference. Existing distributed MAC protocols (typically based on random backoff schemes) are either not resistent to the unpredictable unavailability of the medium at all, or are optimized towards a single network only, in the sense that the nodes of a network collaboratively seek to coordinate the access among themselves [24]. However, the state-of-the-art protocols fail if multiple networks are collocated (as illustrated, for example, in our simulation study in Section 4).

This paper is the first to present (and rigorously prove the performance of) a robust MAC protocol suited for co-existing networks exposed to a harsh environment with unpredictable or even adversarial interference.

### 1.1 Model

We attend to a simplified scenario where a set of $n$ wireless nodes $V$ are located within transmission range of each other and need to communicate over a single shared channel. The wireless nodes belong to $K$ co-existing networks $N_i$ with node sets $V_i$, i.e., $V = V_1 \cup V_2 \cup \ldots \cup V_K$, for some constant $K$ (which is of unknown to the nodes). For simplicity we will assume that these networks are node disjoint. However, by emulating multiple instances, a node may also participate in several networks simultaneously; the performance guarantees derived in this paper would still hold.

We aim to design a distributed MAC protocol for these wireless nodes. Although the protocol is used by all nodes $v \in V$, it should not depend on any knowledge of how many nodes $n$ there are in total, on the number of co-existing networks $K$, or on the size of the co-existing network $v$ belongs to. Moreover, it should ensure that the $K$ networks are independent in the sense that no communication is required between different networks.

Co-existing wireless networks appear in many scenarios where different wireless networks share the same wireless medium. For example, consider a major conference, e.g., organized by the United Nations, where participants from different countries use their hand-held devices to communicate with the other representatives of their country. We assume that the different networks only share the same medium access protocol, but are otherwise different and inter-network communication may not be desired or possible (except, e.g., for multi-national participants). Another scenario where ensuring fairness among co-existing networks is crucial are emergency response networks, where many emergency response


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services, such as fire squads, police, and paramedics, all arrive simultaneously at some accident or disaster scene and have to share the wireless medium in a fair and even manner in order to establish their own separate communication networks.[1]

This paper presents a robust and fair medium access (MAC) protocol COMAC that makes effective use of the few and arbitrarily distributed time periods where a wireless medium is available. We model interference—due to simultaneous transmissions, co-existing networks, changes in the environment that affect the wireless medium, etc., and, when applicable, intentional jamming—generally as an *adversary*, which we may sometimes simply refer to as the *jammer* (even when a malicious jammer is not present in the environment and interference may be caused by other factors). Our adversary may behave in an *adaptive* manner: we assume that the adversary has full knowledge of the protocol and its history, and that it uses this knowledge to decide on whether to jam at a certain moment in time.

Let us use the simplifying notation $N(v)$ to denote the network node $v \in V$ belongs to. We assume that a node $v$ can distinguish among the following events at some time $t$: (1) idle channel (no node in $V$ transmits and there is no outside interference, including jamming activity, at time $t$); (2) successful transmission of a packet in network $N(v)$ (which occurs every time a single node in $N(v)$ transmits, and no other node in $V$ nor the adversary transmits); and (3) medium busy (due to a transmission by a node in some co-existing network different from network $N(v)$, or to simultaneous transmissions by two or more nodes in $N(v)$, or to external interference or jamming).

How to design such a distributed medium access protocol which shares the bandwidth fairly among the $K$ networks, without sacrificing performance? At first sight this may seem impossible: as the total number of co-existing networks and the number of devices is not known, a node cannot guess its fair share of the channel time. This paper shows that this is indeed possible, even in the presence of a powerful adaptive adversarial jammer, referred to as a $(T, 1-\varepsilon)$-*bounded* (adaptive) adversary, which can jam the medium an arbitrary $(1-\varepsilon)$ fraction of the time for an arbitrarily small constant $\varepsilon > 0$ and which hence models a wide range of external interference scenarios or jammers. For the ease of presentation, we assume a synchronous environment where time proceeds in *rounds* (also called *steps*). Formally, the $(T, 1-\varepsilon)$-bounded adversary is defined as follows: for some $T \in \mathbb{N}$ and a constant $0 < \varepsilon < 1$, the adversary may jam at most $(1-\varepsilon)w$ of the time steps, for any time window of size $w \geq T$. In the following, we will use the notation $N = \max\{T, n\}$ to denote the maximum over the adversarial window size and $n$.

Assuming backlogged traffic at the wireless devices, we require that our MAC protocol fulfill the following properties: (1) *c-competitiveness*: Given a time interval $I$, we define $g(I)$ as the number of time steps in $I$ that are non-jammed, and $s(I)$ as the total number of time steps in $I$ in which a successful transmission happens in any network. A MAC protocol is called *c-competitive* against some $(T, 1-\varepsilon)$-bounded adversary if, for any sufficiently large time interval $I$, $s(I) \geq c \cdot g(I)$. (2) *Fairness*: The probabilities of having a successful transmission in any two networks $N_i$ and $N_j$, where $i, j \in [1, K]$, do not differ by much; moreover, the nodes inside a network share the bandwidth fairly as well.

Note that the nodes have no knowledge of how many nodes are there in the same network as itself, nor do the nodes know how many other networks are co-existing and how many nodes are there in each of these co-existing networks, respectively. However, we assume that the nodes have a common parameter $\gamma \in O(1/(\log T + \log\log n))$. The assumption that nodes know $\gamma$ is not critical for the scalability of our protocol, as it requires only a polynomial estimate of $T$ and an even rougher estimate of $n$.

Although the presented COMAC protocol converges fast and is therefore expected to work well under continuously entering and leaving nodes, in this paper we will just focus on a synchronous setting where nodes do not join or leave.

## 1.2 Related Work

The classic approach to design efficient MAC protocols is to use random backoff schemes (e.g., [5, 6, 12, 13, 18, 22]). However, these works do not take into account adversarial interference and are hence not robust against it. Generally, in a random backoff protocol, each node periodically attempts to transmit a message starting with a certain probability $p$. If the message transmission fails (due to interference), the node may retry sending the message in the next time steps with polynomially or exponentially decreasing probabilities (for example, $p^2, p^4, p^8, \ldots$) until the message is successfully transmitted or the minimum allowable probability is reached. Thus, in a dense network (as in our single-hop scenario), an adversary with knowledge of the MAC protocol could simply wait until the nodes have reached transmission probabilities that are inversely proportional to the number of nodes and then start jamming the medium, forcing the nodes to lower their transmission probabilities to a point where a competitive throughput is not achievable.

There also exist several interesting results on protocols that are robust to more complex or even adversarial interference (see, e.g., [7] or [29] for a nice overview). There are two basic approaches in the literature. The first assumes randomly corrupted messages (e.g., [21]), which is much easier to handle than adaptive adversarial jamming [4]. The second line of work either bounds the number of messages that the adversary can transmit or disrupt with a limited energy budget (e.g. [1, 11, 16, 17]), or bounds the number of channels the adversary can jam (e.g. [8, 9, 10, 19]). The protocols in, e.g., [17] can tackle adversarial jamming at both the MAC and network layers, where the adversary may not only jam the channel but also introduce malicious (fake) messages (possibly with address spoofing). However, these solutions depend on the fact that the adversarial jamming budget is finite, so it is not clear whether the protocols would work under heavy continuous jamming. (The result in [11] seems to imply that a jamming rate of 0.5 is the limit whereas the handshaking mechanisms in [17] seem to require an even lower jamming rate.)

Our work is motivated by the jamming-resistant single network MAC protocols studied in [3, 23, 24]. In particular, our adversarial model was introduced by Awerbuch et al. [3] who present a single-hop MAC protocol that guarantees a constant throughput against an adaptive adversary that can block the medium a constant fraction of the time. The MAC protocol and the throughput guarantees were subsequently generalized to multi-hop networks [23, 26], and also the adversary was strengthened further such that it can even jam the medium *reactively*, i.e., it has a binary feedback whether the medium will be idle or busy in the current round [24], before it has to make a decision whether to jam the current round. It has also been shown that the MAC protocol can serve as a basis to design robust applications such as leader election [25].

However, the performance achieved by the MAC protocols described in [3, 23, 24] drops sharply if multiple networks are collo-

[1] Whereas in some scenarios it may be desirable that messages are broadcast across all emergency unit networks, for better immediate response action to a disaster/accident, in the longer run, it is still important to be able to differentiate among the different ad-hoc networks established.

cated. This is due to the fact that in these protocols, each individual co-existing network will strive to achieve a constant competitive throughput in the non-jammed time periods, which requires a constant cumulative access probability *per co-existing network*. As we will explain in the next section in more detail, this necessarily leads to a throughput which is exponentially small in the number of co-existing networks.

It turns out that in a co-existing scenario, the nodes must strike a good balance between a less aggressive (more cooperative) medium access strategy while remaining robust against external interference. We will show that this can be achieved by monitoring the availability of the wireless medium over time and adjusting the sending probabilities or backoffs according to the fraction of observed *idle time periods*. (A similar approach is used in the *Idle-Sense* [14] Distributed Coordination Function to synchronize the nodes' contention windows.) Implicitly synchronizing access via idle time periods is also the key to enable fairness between co-existing networks. The performance analysis of such an algorithm however is involved, as the distributed and randomized decisions exhibit many non-trivial dependencies. Nevertheless, we are able to rigorously prove good competitive throughput and fairness properties, which is also confirmed by our simulation study.

Interestingly, although co-existing networks are ubiquitous and many different aspects are discussed intensively (e.g., the packet inter-arrival time and fairness in co-existing 802.11a/g and 802.11n networks [2], interference cancelation phenomena [27], transmission capacities in multi-antenna adhoc networks [15], or even explicit inter-network communication for frequency cooperation [30]) in different contexts (e.g., in the current debate on white space liberalization [20] where primary TV and microphone users announcing their reservations in a central database are given strict priority), we are not aware of any work on the design of MAC protocols for independent co-existing networks with rigorous formal competitive throughput and fairness guarantees.

## 1.3 Our Contributions

To the best of our knowledge, this is the first paper to present a robust medium access protocol which provably performs well in an environment with co-existing networks. The COMAC protocol features a guaranteed competitive throughput in the presence of co-existing networks as well as a wide range of external interference patterns that can be subsumed and modeled as a $(T, 1-\varepsilon)$-bounded adaptive adversary blocking the medium a $(1-\varepsilon)$ fraction of all time. Moreover, it features fairness among co-existing networks and within an individual network. Finally, the protocol is attractive for its simple design. Our main theoretical result is summarized in the following theorem.

THEOREM 1.1. *The COMAC medium access protocol guarantees that in a backlogged scenario, if executed for* $\Omega(\frac{1}{\varepsilon}\log N \max\{T, \frac{1}{\varepsilon\gamma^2}\log^3 N\})$ *many time steps, COMAC achieves a competitive throughput of* $\Omega(\varepsilon^2 \min\{\varepsilon, 1/poly(K)\})$ *w.h.p., for any* $(T, 1-\varepsilon)$*-bounded adaptive adversary that arbitrarily jams the medium up to a* $(1-\varepsilon)$ *fraction of the time, and which has complete knowledge of the protocol history. Moreover, the cumulative probabilities among different networks, as well as the access probabilities of individual nodes within the same network, differ only by a small factor.*

Simulations complement our theoretical asymptotic bounds.

# 2. MAC FOR CO-EXISTING NETWORKS

Before presenting the formal MAC algorithm, we explain its variables and provide some intuition.

## 2.1 Intuition

In the COMAC protocol, each node $v$ maintains a medium access probability $p_v$ which determines the probability that $v$ transmits a message in a communication round. The nodes adapt and synchronize (inside a co-existing network) their $p_v$ values over time (which as a side-effect also guarantees fairness within the network) in a multiplicative-increase multiplicative-decrease manner in order to ensure a throughput that is as good as possible. More precisely, the sending probabilities are changed by a factor of $(1+\gamma)$. Moreover, we impose an upper bound of $\hat{p}$ on $p_v$, for some constant $0 < \hat{p} < 1$. As we will see, unlike in most classic backoff protocols, our adaptation rules for $p_v$ ensure that the adversary cannot influence $p_v$ much by adaptive jamming.

In addition, each node maintains two variables, a threshold variable $T_v$ and a counter variable $c_v$. $T_v$ is used to estimate the adversary's time window $T$. A good estimation of $T$ can help the nodes recover from a situation where they experience high interference in the network. In times of high interference, $T_v$ will be increased and the sending probability $p_v$ will be decreased.

While these concepts have already been used in our other protocols in [3, 23, 24], they are not sufficient to ensure a jamming-resistant protocol that also works well in case of co-existing networks. The basic problem lies in the fact that all of these protocols aim at reaching a constant cumulative probability, irrespective of the adversarial jamming, so that a good throughput can be obtained in those steps that are not jammed. In co-existing networks, however, this is not a good idea: Suppose that we have $K$ co-existing networks such that each has a constant cumulative probability. Then the overall cumulative probability would be $\Theta(K)$ and therefore, the probability of having a successful transmission in any network would be as low as $\Theta(K)e^{-\Theta(K)}$, which is *exponentially* low in $K$.

Hence, a less aggressive approach than the one pursued in [3, 23, 24] is needed. Ideally, this approach should also make sure that the available bandwidth is shared in a fair way among the networks. Surprisingly, a relatively simple change in the protocol in [24] can achieve jamming-resistance, a good throughput in co-existing networks, and also fairness. The basic idea behind this change is to remember the latest idle time step, and whenever there is a new idle time step, then *with a probability* $q_v$ *that is inversely proportional to the time difference to the previous idle time step*, $p_v$ and $T_v$ are adapted. (The protocol in [24] would *always* adapt $p_v$ and $T_v$ in case of an idle channel.) Since this probabilistic rule turned out to be very hard to analyze, we transformed it into a deterministic rule that shows the same performance in the experiments.

## 2.2 Algorithm

Now we are ready to provide the detailed and formal description of the COMAC algorithm. Initially, each node $v$ sets $p_v = \hat{p}$ ($\hat{p} \leq 1/24$), $c_v = T_v = 1$, and $q_v = 0$. In the following, $L_v \geq 1$ is the time that went by from $v$'s viewpoint since the last idle time step. (If there has not yet been an idle time step, $L_v = \infty$.)

In each step, each node $v$ does the following: $v$ decides with probability $p_v$ to send a message along with the tuple $(c_v, T_v, p_v)$. If it decides not to send a message, it checks the following two conditions:

1. If $v$ senses an idle channel, then $q_v := q_v + 1/L_v$. If $q_v \geq 1$ then
    - $p_v := \min\{(1+\gamma)p_v, \hat{p}\}$, $T_v := \max\{1, T_v - 1\}$, and

- $q_v := q_v - 1$.

2. If $v$ successfully receives a message from node $u$ with the tuple $(c_u, T_u, p_u)$ then

   - $p_v := (1+\gamma)^{-1} p_u$, $c_v = c_u$, and $T_v = T_u$.

Afterwards, $v$ sets $c_v := c_v + 1$. If $c_v > T_v$ then it does the following: $v$ sets $c_v := 1$, and if there was no idle step among the past $T_v$ time steps, then $p_v := (1+\gamma)^{-1} p_v$ and $T_v := T_v + 2$.

# 3. ANALYSIS

For the analysis of our protocol we will use the following notation. We are given $K \geq 2$ co-existing networks denoted by $N_1, \ldots, N_K$. Each network $N_i$ consists of a node set $V_i$ where $n_i = |V_i| \geq 2$ (otherwise, the network would be irrelevant). The cumulative probability due to nodes in $N_i$ is given by $P_i = \sum_{v \in V_i} p_v$, and the cumulative probability over all co-existing networks is given by $P = \sum_{i=1}^{K} P_i$. Whenever we consider some specific time step $t$, $P_i(t)$ is the value of $P_i$ at time $t$ and $P(t)$ is the value of $P$ at time $t$.

## 3.1 Basic Observations

Given that we have a single-hop network, any idle time period is observed by all nodes in all co-existing networks. Hence, the $q_v$ and $L_v$ values of all nodes are identical if all start at the same time (otherwise, two idle time steps suffice to synchronize the $L_v$ values so that the increase of the $q_v$'s is synchronized from that point on, which would also be sufficient for our analysis to go through). Henceforth, we will drop the subscript $v$ from $q_v$ and $L_v$. Since after the first successful transmission in $N_i$, the $T_v$ and $c_v$ values are synchronized among the nodes in $N_i$, we arrive at the following fact, which establishes fairness within a network.

FACT 3.1. *After the first successful transmission in network $N_i$, the access probabilities $p_v$ of the nodes $v \in V_i$ differ by a factor of at most $(1+\gamma)$.*

Throughout our analysis, we will make use of generalized Chernoff bounds that are derived from [28].

LEMMA 3.2. *Consider any set of random variables $X_1, \ldots, X_n$ with values in $[0,1]$. If there exist values $p_1, \ldots, p_n \in [0,1]$ with $\mathbb{E}[\prod_{i \in S} X_i] \leq \prod_{i \in S} p_i$ for every set $S \subseteq \{1, \ldots, n\}$, then it holds for $X = \sum_{i=1}^{n} X_i$ and $\mu = \sum_{i=1}^{n} p_i$ and any $\delta > 0$ that*

$$\mathbb{P}[X \geq (1+\delta)\mu] \leq \left( \frac{e^{\delta}}{(1+\delta)^{1+\delta}} \right)^{\mu} \leq e^{-\frac{\delta^2 \mu}{2(1+\delta/3)}}$$

*If, on the other hand, it holds that $\mathbb{E}[\prod_{i \in S} X_i] \geq \prod_{i \in S} p_i$ for every set $S \subseteq \{1, \ldots, n\}$, then it holds for any $0 < \delta < 1$ that*

$$\mathbb{P}[X \leq (1-\delta)\mu] \leq \left( \frac{e^{-\delta}}{(1-\delta)^{1-\delta}} \right)^{\mu} \leq e^{-\delta^2 \mu/2}$$

The following lemma follows immediately from the Taylor series of the exponential function.

LEMMA 3.3. *For all $0 < x < 1$ it holds that $e^{-x/(1-x)} \leq 1 - x \leq e^{-x}$.*

This implies the following lemma.

LEMMA 3.4. *For any non-jammed time step,*

$$e^{-\frac{P}{1-\hat{p}}} \leq \mathbb{P}[\textit{channel is idle}] \leq e^{-P} \quad \textit{and}$$

$$P_i \cdot e^{-\frac{P}{1-\hat{p}}} \leq \mathbb{P}[\textit{successful msg transmission in } N_i] \leq \frac{P_i}{1-\hat{p}} \cdot e^{-P}$$

## 3.2 Cumulative Probability

In the following, we will derive the first fundamental property of our protocol: we show that the overall cumulative probability $P = \sum_{i=1}^{K} P_i$ converges to some range of values so that the contention on the wireless medium is moderate. This is a necessary condition for a good performance. Our proof framework basically follows the framework of [3] but the proof arguments significantly differ in various places when it comes to analyzing the specifics of our new protocol. We refer to Section 2 of [3] for a comparison.

The proof works by induction over sufficiently large time frames. Let $I$ be a time frame consisting of $\frac{\alpha}{\varepsilon} \log N$ *subframes* $I'$ of size $f = \max\{T, \frac{\alpha\beta^2}{\varepsilon\gamma^2} \log^3 N\}$ rounds, where $\alpha$ and $\beta$ are sufficiently large constants and $N = \max\{T, n\}$. Let $F = \frac{\alpha}{\varepsilon} \log N \cdot f$ denote the size of $I$.

First, we show that for any subframe $I'$ in which initially the overall cumulative probability is at least $1/(f^2(1+\gamma)^{2\sqrt{f}})$, also afterwards this cumulative probability is at least $1/(f^2(1+\gamma)^{2\sqrt{f}})$, w.h.p.

LEMMA 3.5. *For any subframe $I' = [t_0, t_1)$ in which $P(t_0) \geq 1/(f^2(1+\gamma)^{2\sqrt{f}})$, also $P(t_1) \geq 1/(f^2(1+\gamma)^{2\sqrt{f}})$ w.h.p.*

PROOF. We start with the following claim about the maximum number of times nodes decrease their probabilities in $I'$ due to $c_v > T_v$.

CLAIM 3.6. *If in subframe $I'$, $T_v$ is decreased at most $k$ times, then node $v$ increases $T_v$ by 2 at most $k/2 + \sqrt{f}$ many times.*

PROOF. Only an idle time step can potentially reduce $T_v$ by 1. If there is no idle time step during the last $T_v$ many steps, $T_v$ is increased by 2. Suppose that $k = 0$. Then the number of times a node $v$ increases $T_v$ by 2 is upper bounded by the largest possible $\ell$ so that $\sum_{i=0}^{\ell} T_v^0 + 2i \leq f$, where $T_v^0$ is the initial value of $T_v$. For any $T_v^0 \geq 1$, $\ell \leq \sqrt{f}$, so the claim is true for $k = 0$. For each decrease of $T_v$, the current $T_v$ as well as all subsequent values of $T_v$ (until a $T_v$ is reached with $T_v = 1$) get reduced by one. Hence, for an arbitrary value of $k \geq 0$ we are searching for the maximum $\ell$ so that $\sum_{i=0}^{\ell} \max\{T_v^0 + 2i - k, 1\} \leq f$. This $\ell$ is at most $k/2 + \sqrt{f}$, which proves our claim. □

This claim allows us to prove that the overall cumulative probability $P$ will exceed a certain threshold in a subframe w.h.p.

CLAIM 3.7. *Suppose that in $I' = [t_0, t_1)$, $P(t_0) \in [1/(f^2(1+\gamma)^{\sqrt{2f}}), 1/f^2]$. Then there is a time step $t$ in $I'$ with $P(t) \geq 1/f^2$, w.h.p.*

PROOF. Suppose that there are $g$ non-jammed time steps in $I'$. Let $k_0$ be the number of these steps with an idle channel and $k_1$ be the number of these steps with a successful message transmission in any of the co-existing networks. Let the binary random variable $X_i$ be 1 if and only if the nodes increase their access probabilities in the $i$-th idle time step in $I'$, and let $X = \sum_{i=1}^{k_0} X_i$. Furthermore, let $k_2$ be the maximum number of times a node $v$ increases $T_v$ by 2 in $I'$.

Suppose for the moment that $P(t_0) = 1/f^2$. If all time steps $t$ in $I'$ satisfy $P(t) \leq 1/f^2$, then it must hold that the total decrease of $P(t)$ in $I'$ (due to successful transmissions and cases in which access probabilities are decreased when $c_v > T_v$), which is at most $(1+\gamma)^{k_1+k_2}$, has to be at least as large as the total increase of $P(t)$ (due to idle time steps), which is equal to $(1+\gamma)^X$. Hence, we must have that $X \leq k_1 + k_2$. For an arbitrary initial probability $P(t_0) \leq 1/f^2$, we must therefore have

$$X - \log_{1+\gamma}((1/f^2)/P(t_0)) \leq k_1 + k_2 \tag{1}$$

to avoid a time step $t$ in $I'$ with $P(t) > 1/f^2$. Our goal is to show that this inequality is violated w.h.p., which implies that $I'$ has a time step $t$ with $P(t) > 1/f^2$ w.h.p.

Next, we focus on $k_2$. Consider some fixed $k_0 \geq 2$ (as we will see later, $k_0 \geq 2$ w.h.p.). Let $L_i$ be the $L$-value of the nodes at the $i$-th idle time step (note that they are all the same) and let $q_i = 1/L_i$ denote the increase of the $q$-values of the nodes in the $i$-th idle time step. Also, let $\bar{q} = \frac{1}{k_0-1}\sum_{i=2}^{k_0} q_i$. Certainly, the number of times any node $v$ decreases $T_v$ in $I'$ is bounded by the number of times $q$ is at least 1, which is at most $\lceil \sum_{i=1}^{k_0} q_i \rceil \leq \lceil 1 + (k_0-1)\bar{q} \rceil$. Hence, it follows from Claim 3.6 that

$$k_2 \leq \lceil \bar{q}(k_0-1)+1 \rceil/2 + \sqrt{f} \tag{2}$$

On the other hand, the number of times any node $v$ increases $p_v$ in $I'$ is at least $\lfloor \sum_{i=2}^{k_0} q_i \rfloor = \lfloor (k_0-1)\bar{q} \rfloor$ (because due to Fact 3.1 it follows from $P(t) \leq 1/f^2$ that $p_v(t) < \hat{p}$ for all $v$). Plugging this together with (2) into (1) and using the fact that $P(t_0) \geq 1/(f^2(1+\gamma)^{\sqrt{2f}})$, we obtain

$$\begin{aligned} \lfloor (k_0-1)\bar{q} \rfloor - \lceil (k_0-1)\bar{q}+1 \rceil/2 &\leq \sqrt{2f} + k_1 + \sqrt{f} \\ \Rightarrow \quad (k_0-1)\bar{q}/2 &\leq k_1 + 4\sqrt{f} \end{aligned} \tag{3}$$

given that $f$ is large enough. It remains to lower bound $\bar{q}$ and $k_0$ and to upper bound $k_1$ in order to arrive at a contradiction.

We start with $\bar{q}$. Let $\bar{L} = \frac{1}{k_0-1}\sum_{i=2}^{k_0} L_i$. Since $\sum_{i=2}^{k_0} L_i < f$, it holds that $\bar{L} < \frac{f}{k_0-1}$. Moreover, we make use of the following well-known fact.

FACT 3.8. *For any sequence of positive numbers $x_1, \ldots, x_n$ it holds for its arithmetic mean $A = (1/n)\sum_{i=1}^n x_i$ and its harmonic mean $H = ((1/n)\sum_{i=1}^n 1/x_i)^{-1}$ that $A \geq H$.*

Hence, it follows that $\bar{L} \geq 1/(\frac{1}{k_0-1}\sum_{i=2}^{k_0} 1/L_i)$ and therefore, $\frac{1}{k_0-1}\sum_{i=2}^{k_0} 1/L_i \geq 1/\bar{L}$. This in turn implies that

$$\bar{q} \geq 1/\bar{L} \geq \frac{k_0-1}{f}$$

Next we provide an upper bound for $k_1$ that holds w.h.p. Certainly, for any time step $t$ with $P(t) \leq 1/f^2$,

$$\mathbb{P}[\geq 1 \text{ message transmitted at step } t] \quad \leq \quad 1/f^2.$$

Hence, $\mathbb{E}[k_1] \leq g \cdot (1/f^2) \leq 1/f$. In order to prove an upper bound on $k_1$ that holds w.h.p., we can use the general Chernoff bounds stated in Lemma 3.2. For any step $t$ let the binary random variable $Y_t$ be 1 if and only if at least one message is transmitted successfully at time $t$ and $P(t) \leq 1/f^2$. Then

$$\begin{aligned} \mathbb{P}[Y_t = 1] &= \mathbb{P}[P(t) \leq 1/f^2] \cdot \\ &\quad \mathbb{P}[\text{successful msg transmission} \mid P(t) \leq 1/f^2] \\ &\leq 1/f^2. \end{aligned}$$

Moreover, it certainly holds for any set $S$ of time steps prior to some time step $t$ that

$$\mathbb{P}[Y_t = 1 \mid \prod_{s \in S} Y_s = 1] \leq 1/f^2.$$

Therefore, we have

$$\begin{aligned} &\mathbb{P}[\prod_{s \in S} Y_s = 1] \\ &= \mathbb{P}[Y_1 = 1] \cdot \mathbb{P}[Y_2 = 1 | Y_1 = 1] \cdot \mathbb{P}[Y_3 = 1 | \prod_{s=1,2} Y_s = 1] \cdot \ldots \\ &\quad \cdot \mathbb{P}[Y_{|S|} = 1 | \prod_{s=1,2,\ldots,|S|-1} Y_s = 1] \\ &\leq (1/f^2)^{|S|} \end{aligned}$$

and

$$\mathbb{E}[\prod_{s \in S} Y_s = 1] = \mathbb{P}[\prod_{s \in S} Y_s = 1] \leq (1/f^2)^{|S|}.$$

Thus, the Chernoff bounds and our choice of $f$ imply that w.h.p. either $\sum_{t \in I'} Y_t < \varepsilon^2 f/8$ and $P(t) \leq 1/f^2$ throughout $I'$, or there must be a time step $t$ in $I'$ with $P(t) > 1/f^2$, which would finish the proof. Therefore, unless $P(t) > 1/f^2$ at some point in $I'$, $k_1 < \varepsilon^2 f/8$ w.h.p.

Next we prove a lower bound on $k_0$ that holds w.h.p. For any time step $t$ with $P(t) \leq 1/f^2$ it holds that

$$\mathbb{P}[\text{channel is idle}] \geq e^{-P(t)/(1-\hat{p})} \geq 1 - \frac{P(t)}{1-\hat{p}} \geq 1 - 1/f$$

Hence, $\mathbb{E}[k_0] \geq g \cdot (1-1/f) \geq \varepsilon f(1-1/f)$. Using similar arguments as for $k_1$, it follows that $k_0 > (7/8)\varepsilon f$ w.h.p. unless $P(t) > 1/f^2$ at some point in $I'$. When combining the bounds for $\bar{q}$ and $k_0$, we obtain

$$\begin{aligned} (k_0-1)\bar{q}/2 &\geq \frac{(k_0-1)^2}{2f} \geq (7/8)^2\varepsilon^2 f/2 \\ &> \varepsilon^2 f/8 + 4\sqrt{f} > k_1 + 4\sqrt{f} \end{aligned}$$

w.h.p., if $f$ is large enough, which violates Inequality (3) and therefore completes the proof of Claim 3.7. □

Similarly, we can also prove that once the cumulative probability exceeds a certain threshold, it cannot become too small again.

CLAIM 3.9. *Suppose that for the first time step $t_0$ in $I'$, $P(t_0) \geq 1/f^2$. Then there is no time step $t$ in $I'$ with $P(t) < \frac{1}{f^2(1+\gamma)^{\sqrt{2f}}}$, w.h.p.*

PROOF. Consider some fixed subinterval $I'' = [t_1, t_2)$ in $I'$ with the property that $P(t_1) \geq 1/f^2$ and $P(t) \leq 1/f^2$ for all other $t$ in $I''$ (i.e., we will use conditional probabilities based on $P(t) \leq 1/f^2$ like in the bound for $k_1$ in the proof of Claim 3.7). Suppose that there are $g$ non-jammed time steps in $I''$. If $g \leq \beta \log N$ for a (sufficiently large) constant $\beta$, then it follows for the probability $P(t_2)$ at the end of $I''$ that

$$P(t_2) \geq \frac{1}{f^2} \cdot (1+\gamma)^{-((3/2)\beta \log N + \sqrt{f})} \geq \frac{1}{f^2(1+\gamma)^{\sqrt{2f}}}$$

given that $f$ is large enough (i.e., $\varepsilon = \Omega(1/\log^3 N)$). This is because in the worst case for the decrease of $P(t)$ all non-jammed time steps are successful. In this case, $P(t)$ is decreased at most

$\beta \log N$ times due to these steps. Moreover, from Claim 3.6 it follows that $P(t)$ can be decreased another at most $\beta \log N/2 + \sqrt{f}$ times due to $c_v > T_v$.

So suppose that $g > \beta \log N$. Let $X$ be the number of time steps in $I''$ in which $P(t)$ increases and $k_1$ be the maximum number of time steps in $I''$ (over all networks) with a successful message transmission. Furthermore, let $k_2$ be the maximum number of times a node $v$ increases $T_v$ in $I''$. If $P(t_2) < \frac{1}{f^2(1+\gamma)^{\sqrt{2f}}}$ then it must hold that the total increase in $P(t)$ (which is equal to $(1+\gamma)^X$) is at most the total decrease in $P(t)$ (which is at most $(1+\gamma)^{k_1+k_2}$), or in other words,

$$X \leq k_1 + k_2.$$

From the previous claim we know that this is not true w.h.p. given that $P(t) \leq 1/f^2$ for all $t > t_1$ in $I''$ and the constant $\beta$ is sufficiently large to achieve polynomially small probability bounds. Since there are at most $f^2$ possible values for $t_1$ and $t_2$, there is no time step $t_2$ in $I'$ with $P(t_2) < \frac{1}{f^2(1+\gamma)^{\sqrt{2f}}}$ w.h.p., which completes the proof. $\square$

Combining Claims 3.7 and 3.9 completes the proof of Lemma 3.5. $\square$

Next we show an upper bound for $P(t)$. In the following, $K' = O(K)$ is a sufficiently large constant $\geq K$.

LEMMA 3.10. *For any subframe $I' = [t_0, t_1)$ with $T_v \leq (3/4)\sqrt{F}$ for all nodes $v$ at the beginning of $I'$, $P(t_1) \leq 12 \ln K'$ w.m.p.*

PROOF. First, we will show that if $P(t) \geq 4 \ln K'$ throughout $I'$, then for each $N_i$, there must be a step $t'$ with $P_i(t') \leq (2 \ln K')/K'$ w.h.p., and once such a step is reached, we show that $P_i(t'') < (4 \ln K')/K'$ w.m.p. for all time steps $t''$ following $t'$. Hence, there must be a time step $t''$ in $I'$ with $P_i(t'') < (4 \ln K')/K'$ for all $i$, w.m.p., contradicting the assumption that $P(t) \geq 4 \ln K'$ throughout $I'$. Once we have that, we will show that at the end of $I'$, $P(t_1) \leq 12 \ln K'$ w.m.p.

Consider some fixed network $i$. Let $k_0$ be the number of idle steps in $I'$ and $k_1$ be the number of successful time steps for network $i$. Moreover, let $X$ be the total number of times $P_i(t)$ is increased by $(1+\gamma)$ due to an idle channel in $I'$. For $N_i$ to avoid a time step $t'$ in $I'$ with $P_i(t') \leq (2 \ln K')/K'$, we must have that the total increase of $P_i(t)$ (which is equal to $(1+\gamma)^X$) is at least the total decrease of $P_i(t)$ once we have reached a point $t$ with $P_i(t) = (2 \ln K')/K'$, which is the case after at most $\log_{1+\gamma}(n_i \cdot \hat{p})$ reductions of $P_i(t)$. Hence, we must have

$$X \quad \geq \quad k_1' - \log_{1+\gamma}(n_i \cdot \hat{p}) \tag{4}$$

where $k_1'$ is the total decrease (in the exponent) of $P_i(t)$ due to successful transmissions to avoid a time step $t'$ in $I'$ with $P_i(t') \leq (2 \ln K')/K'$. Notice that $k_1'$ is not equal to $k_1$ because if, for example, a node successfully transmits twice in a row, $P_i(t)$ does not get decreased the second time.

In order to contradict this bound, we first need to have a closer look at what happens when there is a successful transmission in $N_i$.

CLAIM 3.11. *If the node $v$ successfully transmitting a message in $N_i$ at time $t$ is different from the node that previously successfully transmitted a message in $N_i$, then $P_i(t+1) \in [\frac{1}{1+\gamma} P_i(t), \frac{1}{\sqrt{1+\gamma}} P_i(t)]$ for any $n_i \geq 2$.*

PROOF. The lower bound is obvious. Moreover, it follows from the protocol that

$$\begin{aligned}
P_i(t+1) &= p_{v,t} + \sum_{w \in V_i \setminus \{v\}} \frac{1}{1+\gamma} \cdot p_{v,t} \\
&= \frac{1}{1+\gamma} \cdot P_i(t) + \frac{\gamma}{1+\gamma} \cdot p_{v,t} \\
&\leq \frac{1}{1+\gamma} \cdot P_i(t) + \frac{\gamma}{1+\gamma} \cdot \frac{P_i(t)}{n_i} \\
&= \frac{1}{1+\gamma} \left(1 + \frac{\gamma}{n_i}\right) P_i(t) \\
&\leq \frac{1}{1+\gamma} (1+\gamma)^{1/n_i} P_i(t) \leq \frac{1}{\sqrt{1+\gamma}} P_i(t)
\end{aligned}$$

given that $n_i \geq 2$. $\square$

If the same node $v$ successfully transmits again at time $t$, then $P_i(t+1) = P_i(t)$, which only happens with probability at most $(1+\gamma)/n_i$ because in this case the transmitting node has an access probability that is by a $(1+\gamma)$ factor larger than the other access probabilities in $N_i$. Hence, on expectation, at least $1/3$ of the time steps with successful transmission, $P_i(t)$ is reduced by at least $(1+\gamma)^{1/2}$, which implies that $\mathbb{E}[k_1'] \geq k_1/6$.

Based on this insight, the next claim shows that under certain conditions, Inequality (4) is not true w.h.p. Let $g_i$ be the number of *useful* time steps for $N_i$, which are time steps that are either idle or successful for $N_i$ in $I'$.

CLAIM 3.12. *If all time steps $t \in I'$ satisfy $P(t) \geq 4 \ln K'$ and $g_i \geq \delta \log_{1+\gamma} N$ for a sufficiently large constant $\delta$, then $X + \log_{1+\gamma} n_i < k_1'$ w.h.p.*

PROOF. It is easy to see that for any useful time step $t$,

$$\mathbb{P}[t \text{ successful for } N_i] \quad \geq \quad P_i(t) \cdot \mathbb{P}[t \text{ idle}] \tag{5}$$

and therefore $\mathbb{E}[k_1] \geq \frac{2 \ln K'}{K'} \mathbb{E}[k_0]$ unless there is a time step $t$ with $P_i(t) < (2 \ln K')/K'$. For a given number of useful time steps $g_i$, since $k_0 + k_1 = g_i$ and therefore also $\mathbb{E}[k_0] + \mathbb{E}[k_1] = g_i$, $\mathbb{E}[k_1] \geq \frac{2 \ln K'}{K'}(g_i - \mathbb{E}[k_1])$, which implies that $\mathbb{E}[k_1] \geq \frac{\ln K'}{K'} \cdot g_i$ if $K' = O(K)$ is a sufficiently large constant. Since $\mathbb{E}[k_1'] \geq k_1/6$, $g_i = \Omega(\log_{1+\gamma} N)$, and for each useful time step there is an independent probability whether this time step is idle or successful, it follows from the Chernoff bounds that $k_1' \geq \frac{\ln K'}{8K'} g_i$ w.h.p.

Next we bound $X$. Let the binary random variable $X_j$ denote the increase of $P_i(t)$ by $(1+\gamma)^{X_j}$ in the $j$-th idle time step. Then $X = \sum_{j=1}^{k_0} X_j$. Moreover, let $L_j$ be the number of time steps between the $(j-1)$-th and $j$-th idle time steps. It holds that

$$\mathbb{P}[t \text{ idle}] \leq e^{-P(t)} \leq 1/(K')^4$$

for every $t \in I'$ given that $P(t) \geq 4 \ln K'$. Hence,

$$\begin{aligned}
\mathbb{E}[X_j] &= \sum_{\ell \geq 1} \mathbb{P}[L_j = \ell] \cdot 1/\ell \leq \sum_{\ell \geq 1} \frac{1}{(K')^4} \left(1 - \frac{1}{(K')^4}\right)^{\ell-1} \cdot \frac{1}{\ell} \\
&\leq \frac{1}{(K')^4 - 1} \sum_{\ell \geq 1} e^{-\ell/(K')^4}/\ell \leq \frac{1}{(K')^4 - 1} \cdot 2 \ln (K')^4 \\
&= \frac{4 \ln K'}{(K')^4 - 1}
\end{aligned}$$

and therefore, $\mathbb{E}[X] \leq \frac{4 \ln K'}{(K')^4 - 1} \cdot k_0 \leq \frac{4 \ln K'}{(K')^4 - 1} \cdot g_i$. Since the upper bound on $\mathbb{E}[X_j]$ holds independently for each $j$, it follows from the Chernoff bounds that $X \leq \frac{6 \ln K'}{(K')^4} \cdot g_i$ w.h.p.

Since $g_i = \Omega(\log_{1+\gamma} N)$, $X + \log_{1+\gamma} n_i < k_1'$ w.h.p. if $K' = O(K)$ is sufficiently large, which completes the proof of the claim. $\square$

Otherwise, suppose that $g_i < \delta \log_{1+\gamma} N$. For every node $v$ it follows from the COMAC protocol and the choice of $f$ and $F$ that if initially $T_v \leq (3/4)\sqrt{F}$, then $T_v$ can be at most $\sqrt{F}$ during $I'$. Let us cut $I'$ into $m$ intervals of size $2\sqrt{F}$ each. It is easy to check that if $\beta$ in the definition of $f$ is sufficiently large compared to $\delta$, then $m \geq 3\delta \log_{1+\gamma} N$. Since there are less than $\delta \log_{1+\gamma} N$ useful steps in $N_i$ in $I'$, at least $2\delta \log_{1+\gamma} N$ of these intervals do not contain any useful step, which implies that $p_v$ is reduced by $(1+\gamma)$ by each $v \in V_i$ in each of these intervals.

Hence, altogether, every $p_v$ gets reduced by a factor of at least $(1+\gamma)^{-2\delta \log_{1+\gamma} N}$ during $I'$ in $N_i$. The useful time steps can only raise that by at most $(1+\gamma)^{\delta \log_{1+\gamma} N}$, so altogether we must have $P_i(t') \leq (2 \ln K')/K'$ at some time point $t'$ in $I'$, w.h.p.

Next we prove the following claim, which implies that for all $t'' > t'$ in $I'$, $P_i(t'') < (4 \ln K')/K'$ w.m.p.

CLAIM 3.13. *If all time steps $t \in I'$ satisfy $P(t) \geq 4 \ln K'$ and initially $P_i(t) \leq (2 \ln K')/K'$, then for all steps $t \in I'$, $P_i(t) \leq (4 \ln K')/K'$ w.m.p.*

PROOF. Consider some fixed subinterval $I'' = [t_1, t_2)$ in $I'$ with the property that $P_i(t_1) \leq (2 \ln K')/K'$ and $P_i(t) \geq (2 \ln K')/K'$ for all other $t$ in $I''$. Suppose that there are $g_i$ useful time steps in $I''$. If $g_i \leq \ln_{1+\gamma} 2$, then it follows for the probability $P_i(t_2)$ at the end of $I''$ that $P_i(t_2) \leq \frac{2 \ln K'}{K'} \cdot (1+\gamma)^{\ln_{1+\gamma} 2} \leq \frac{4 \ln K'}{K'}$. Otherwise, suppose that $g_i > \ln_{1+\gamma} 2$, which is at least $1/(2\gamma) = \Omega(\ln f)$. Let $X$ be the number of time steps in $I''$ in which $P_i(t)$ increases and $k_1$ be the number of time steps in $I''$ with a successful transmission in $N_i$. Furthermore, let $k_2$ be the maximum number of times a node $v \in V_i$ increases $T_v$ in $I''$. If $P(t_2) > (4 \ln K')/K'$ then it must hold that the total increase in $P_i(t)$ (which is equal to $(1+\gamma)^X$) is at least the total decrease in $P(t)$ (which is at most $(1+\gamma)^{k_1+k_2}$) plus $\ln_{1+\gamma} 2$, or formally,

$$X \geq k_1' + \ln_{1+\gamma} 2 \tag{6}$$

where $k_1'$ is the total decrease (in the exponent) of $P_i(t)$ due to successful transmissions. We know that $\mathbb{E}[k_1'] \geq k_1/6$. Also, from the proof of the previous claim it follows that $\mathbb{E}[k_1] \geq \frac{\ln K'}{K'} g_i$ if $K' = O(K)$ is a sufficiently large constant, unless there is a time step $t$ in $I'$ with $P_i(t) < (2 \ln K')/K'$. Since $g_i = \Omega(\ln f)$, it follows from the Chernoff bounds that $k_1' \geq \frac{\ln K'}{8K'} g_i$ w.m.p. On the other hand, it follows from the proof of the previous claim that $X \leq \frac{6 \ln K'}{(K')^4} \cdot g_i$ w.m.p. Hence, inequality (6) is violated w.m.p., which implies that $P_i(t_2) \leq \frac{4 \ln K'}{K'}$ w.m.p. Since there are at most $f^2$ different values of $t_1$ and $t_2$, there is no time step $t_2$ in $I'$ with $P_i(t_2) > \frac{4 \ln K'}{K'}$ w.m.p., which completes the proof. $\square$

Combining the insights above, it follows that there must be a time step $t$ in $I'$ with $P(t) < 4 \ln K'$ w.m.p. To finish the proof, we need the following claim.

CLAIM 3.14. *If for the first time step $t_0$ in $I'$, $P(t_0) \leq 4 \ln K'$, then $P(t) \leq 12 \ln K'$ for all time steps $t$ in $I'$ w.m.p.*

PROOF. Consider some subinterval $I'' = [t_1, t_2)$ in $I'$ with the property that $P(t_1) \leq 4 \ln K'$ and $P(t) \geq 4 \ln K'$ for all $t > t_1$ in $I''$. Suppose that there are $g$ useful time steps in $I''$, where a time step is useful if there was either a successful transmission in some network or the channel is idle. If $g \leq \log_{1+\gamma} 2$, then certainly $P(t) \leq 12 \ln K'$ for all $t$ in $I'$. So suppose that $g > \log_{1+\gamma} 2$. Consider some fixed network $N_i$. Let $X$ be the number of time steps in $I''$ in which $P_i(t)$ increases and $k_1$ be the number of time steps in $I''$ with a successful message transmission in $N_i$. Furthermore, let $k_2$ be the maximum number of times a node $v \in V_i$ increases $T_v$ in $I''$. If $P(t_2) > 12 \ln K'$ then there must be a network $N_i$ with $P_i(t_2) > \max\{(8 \ln K')/K', 2P_i(t_1)\}$. To see this, let $I_1$ be the set of all $i$ with $P_i(t_1) < (4 \ln K')/K'$ and $I_2$ be the set of all other $i$. As long as for all $i$, $P_i(t_2) \leq \max\{(8 \ln K')/K', 2P_i(t_1)\}$, it must hold that $P(t_2) \leq \sum_{i \in I_1} (8 \ln K')/K' + \sum_{i \in I_2} 2P_i(t_1) \leq (8 \ln K')/K' \cdot K + 2P(t_1) \leq 12 \ln K'$ if $K' = O(K)$ is sufficiently large.

First, consider the case that for some $i$ with $P_i(t_1) \geq (4 ln K')/K'$, $P_i(t_2) > 2P_i(t_1)$. Then the total increase of $P_i(t)$ in $I''$ (which is equal to $(1+\gamma)^X$ is at least the total decrease in $P_i(t)$ plus $\log_{1+\gamma} 2$. Hence,

$$X \geq k_1' + \log_{1+\gamma} 2 \tag{7}$$

where $k_1'$ is the total decrease (in the exponent) of $P(t)$ due to successful transmissions in $N_i$. From Inequality (5) we know that $\mathbb{E}[k_1] \geq \frac{4 \ln K'}{K'} \cdot \mathbb{E}[k_0]$ and therefore $\mathbb{E}[k_1] \geq \frac{2 \ln K'}{K'} \cdot g$ if $K' = O(K)$ is large enough. Since $\mathbb{E}[k_1'] \geq k_1/6$ and $g = \Omega(\ln f)$ it follows from the Chernoff bounds that $k_1' \geq \frac{\ln K'}{4K'} \cdot g$ w.m.p. On the other hand, we also know that $X \leq \frac{6 \ln K'}{(K')^4} \cdot g$ w.m.p., which implies that Inequality (7) is violated w.m.p. Hence, $P_i(t_2) \leq 2P_i(t_1)$ w.m.p.

For the case that $P_i(t_1) < (4 \ln K')/K'$ let $t_1'$ be the first step in $I''$ with $P_i(t_1') \geq (4 \ln K')/K'$. If $t_1'$ does not exist, we are done, and otherwise we prove in the same way as above that w.m.p. $P_i(t_2) \leq (12 \ln K')/K'$.

Since there are at most $f^2$ ways of choosing $t_1$ and $t_2$, there is no time step $t$ in $I'$ with $P(t) \leq 12 \ln K'$ w.m.p., which completes the proof. $\square$

All claims combined imply Lemma 3.10. $\square$

A proof similar to Lemma 3.10 also implies the following result.

COROLLARY 3.15. *For any subframe $I'$ that satisfies $P(t) \leq 12 \ln K'$ at the beginning of $I'$, all time steps $t$ of $I'$ satisfy $P(t) \leq 36 \ln K'$ w.m.p.*

We also need to show that for a constant fraction of the non-jammed time steps in a subframe where initially $P(t) \leq 12 \ln K'$, $P(t)$ is also lower bounded by a constant for a sufficiently large fraction of time steps $t$.

LEMMA 3.16. *For any subframe $I'$ in which initially $P(t_0) \geq 1/(f^2(1+\gamma)^{2\sqrt{f}})$, at least $\varepsilon/8$ of the non-jammed steps $t$ satisfy $P(t) \geq \varepsilon\hat{p}/4$, w.h.p.*

PROOF. Let $G$ be the set of all non-jammed time steps in $I'$ and $S$ be the set of all steps $t$ in $G$ with $P(t) < \varepsilon\hat{p}/4$. Let $g = |G|$ and $s = |S|$. If $s \leq (1-\varepsilon/8)g$, we are done. Hence, consider the case that $s \geq (1-\varepsilon/8)g$.

Suppose that $P(t)$ must be increased $\ell$ many times to get from its initial value up to a value of $\varepsilon\hat{p}/4$. (If $P(t_0) \geq \varepsilon\hat{p}/4$ then $\ell = 0$.) Let $k_0$ be the number of time steps in $S$ with an idle channel and $k_1$ be the number of time steps in $S$ with a successful message transmission in any of the co-existing networks. Let the binary random variable $X_i$ be 1 if and only if the nodes increase their access probabilities in the $i$-th idle time step in $S$, and let

$X = \sum_{i=1}^{\ell} X_i$. Furthermore, let $k_2$ be the maximum number of times a node $v$ decreases $p_v$ due to $c_v > T_v$ in $I'$. For $S$ to be feasible (i.e., probabilities can be assigned to each $t \in S$ so that $P(t) < \varepsilon\hat{p}/4$), we must have

$$X \leq \ell + k_1 + k_2 \tag{8}$$

For the special case that $\ell = k_2 = 0$ this follows from the fact that whenever there is a successful message transmission, $P(t)$ is reduced by $(1+\gamma)^{-1}$, at most. On the other hand, whenever the nodes decide to increase $P(t)$ for some $t \in S$, $P(t)$ can indeed increase because of $P(t) < \varepsilon\hat{p}/4$ and therefore $p_v < \hat{p}$ for all $v$. Thus, if $X > k_1$, then one of the steps in $S$ would have to have a probability of at least $\varepsilon\hat{p}/4$, violating the definition of $S$. $\ell$ comes into the formula due to the startup cost of getting to a value of $\varepsilon\hat{p}/4$, and $k_2$ comes into the formula since the reductions of the $p_v(t)$ values due to $c_v > T_v$ allow up to $k_2$ additional decreases of $P(t)$ for $S$ to stay feasible.

Certainly, $\ell \leq 2\log_{1+\gamma} f + 2\sqrt{f}$. Moreover, for $k_1$ it holds that $\mathbb{E}[k_1] \leq \varepsilon\hat{p}/4 \cdot s$ and therefore, $k_1 \leq \varepsilon\hat{p}/2 \cdot s$ w.h.p. For $k_2$ it holds that $k_2 \leq (X + \varepsilon g/8)/2 + \sqrt{f}$. Hence, Inequality (8) implies that

$$\begin{aligned} X &\leq 2\log_{1+\gamma} f + 2\sqrt{f} + \varepsilon\hat{p}s/2 + (X + \varepsilon g/8)/2 + \sqrt{f} \\ \Rightarrow \quad X &\leq (\hat{p} + 1/16)\varepsilon g + 8\sqrt{f} \end{aligned} \tag{9}$$

if $f$ is sufficiently large. It remains to compute a lower bound for $X$.

Let $X'$ be the total number of times $P(t)$ is increased over all time steps in $G$, $k_0'$ be the number of idle time steps in $G$, and $\bar{q}$ be the average increase of the $q_v$-values in $I'$. From the proof of Claim 3.7 we know that $\bar{q} \geq (k_0'-1)/f$ and that $X' \geq \lfloor (k_0'-1)\bar{q} \rfloor$. Moreover, $X \geq X' - \varepsilon g/8$. Hence, $X \geq \lfloor (k_0-1)^2/f \rfloor - \varepsilon g/8$. We know that $\mathbb{E}[k_0] \geq (1 - \varepsilon\hat{p}/4)s$ and therefore, $k_0 \geq 3g/4$ w.h.p. Hence, $X \geq g^2/(4f) - \varepsilon g/8 \geq \varepsilon g/8$ w.h.p. Since this violates Inequality (9), the lemma follows. □

In the following, let us call a subframe $I'$ *good* if its initial step $t_0$ satisfies $P(t_0) \leq 12 \ln K'$. Combining the results above, we get:

LEMMA 3.17. *For any good subframe $I'$, there are at least $\varepsilon^2 f/8$ non-jammed time steps $t$ in $I'$ with $P(t) \in [\varepsilon\hat{p}/4, 36\ln K']$ w.m.p.*

Consider now the first eighth of frame $I$, called $J$. The following lemma follows directly from Lemma 2.14 in [3].

LEMMA 3.18. *If at the beginning of $J$, $p_v \geq 1/(f^2(1+\gamma)^{2\sqrt{f}})$ and $T_v \leq \sqrt{F}/2$ for all nodes $v$, then we also have $p_v \geq 1/(f^2(1+\gamma)^{2\sqrt{f}})$ at the end of $J$ for every $v$ and the number of non-jammed time steps $t$ in $I'$ with $P(t) \in [\varepsilon\hat{p}/4, 36\ln K']$ is at least $\varepsilon^2 f/16$ w.h.p.*

We finally need the following lemma, which follows from Lemma 2.15 in [3].

LEMMA 3.19. *If at the beginning of $J$, $T_v \leq \sqrt{F}/2$ for all $v$, then it holds that also $T_v \leq \sqrt{F}/2$ at the end of $J$ w.h.p.*

Inductively using Lemmas 3.18 and 3.19 on the eighths of frame $I$ implies that COMAC satisfies the property of Lemmas 3.18 for the entire $I$ and at the end of $I$, $p_v \geq 1/(f^2(1+\gamma)^{2\sqrt{f}})$ and $T_v \leq \sqrt{F}/2$ for all $v$ w.h.p. Since our results hold with high probability, we can also extend them to any polynomial number of frames.

## 3.3 Throughput

Summarizing the results above, we obtain the following result for the throughput.

THEOREM 3.20. *For any polynomial sequence of time steps of length at least $F$,* COMAC *achieves a competitive throughput of $\Omega(\varepsilon^2 \min\{\varepsilon, 1/poly(K)\})$ for any constants $\varepsilon$ and $K$.*

## 3.4 Fairness

Finally, we show that COMAC also ensures a limited degree of fairness. Note that by Lemma 3.4, we can directly bound the probabilities of having a successful transmission within networks $N_i$ and $N_j$ by their respective cumulative probabilities, which we bound on the following theorem.

THEOREM 3.21. *If all nodes $v$ initially start with access probability $\hat{p}$, then it takes at most $F$ time steps until a time step is reached in which the difference between minimum and maximum cumulative probability of a network is at most $O(K^2)$.*

PROOF. Consider the potential function summing up the differences of the networks' cumulative probabilities compared to the minimum probability $\Phi = \sum_i |x_i - x_{\min}|$ where $x_i = \log_{1+\gamma} P_i$ and $x_{\min} = \min_i x_i$. We focus on the events with a successful transmission, since only successful transmissions can change the difference among individual network probabilities. Assume that a successful tranmission occured in $N_i$, if $x_i > x_{\min}$, then the change in $\Phi$, denoted by $\Delta\Phi$, satisfies $\Delta\Phi = -1$. If $x_i = x_{\min}$, then $\Delta\Phi \leq K$. Hence, $\mathbb{E}[\Delta\Phi] \leq -\mathbb{P}[x_i > x_{\min}$ successful$] + K\mathbb{P}[x_i = x_{\min}$ successful$]$. Suppose that $x_{\max} \geq x_{\min} + \log_{1+\gamma}(2K^2)$. Then, $\mathbb{P}[x_i > x_{\min}$ successful$] \geq 2K \cdot \mathbb{P}[x_i = x_{\min}$ successful$]$ as there can be up to $K-1$ many $N_i$ with $x_i = x_{\min}$. Certainly, $\mathbb{P}[x_i > x_{\min}$ successful$] + \mathbb{P}[x_i = x_{\min}$ successful$] = 1$ given that there is a successful transmission. Hence in this case, $\mathbb{P}[x_i > x_{\min}$ successful$] \geq \frac{2K}{2K+1}$, which implies that $\mathbb{E}[\Delta\Phi] \leq -\frac{2K}{2K+1} + \frac{K}{2K+1} = -\frac{K}{2K+1} \leq -1/3$, whenever there is a successful transmission.

Now, let us define the random variable $X_t$ as follows for the $t$-th successful transmission: $X_t = 1$ if either $x_{\max} < x_{\min} + \log_{1+\gamma}(2K^2)$ (i.e., we reached our goal) or the successful transmission is from a network $N_i$ with $x_i > x_{\min}$; and $X_t = -K$ otherwise.

Suppose that there are $s$ successful transmissions across all networks. Let $X = \sum_{t=1}^{s} X_t$. Then it holds that $\mathbb{E}[X] \geq s/3$. In order to apply Chernoff bounds, let us define $Y_t = (X_t + K)/(K+1)$ and $Y = \sum_{t=1}^{s} Y_t$. Then $Y_t$ is a binary random variable with $\mathbb{E}[Y_t] \geq (K+1/3)/(K+1)$ and therefore $\mathbb{E}[Y] \geq s(K+1/3)/(K+1)$. Since the upper bound on $\mathbb{E}[Y_t]$ holds irrespective of previous $Y_j$'s, it follows from the Chernoff bounds that $\mathbb{P}[Y \leq (1-\delta)s(K+1/3)/(K+1)] \leq e^{-\delta^2 s/3}$, for any $0 < \delta < 1$. Since $Y = (X + s \cdot K)/(K+1)$, we get $\mathbb{P}[X \leq (1-\delta)s/3 - \delta sK] \leq e^{-\delta^2 s/3}$. If we choose $\delta = 1/(6(K+1/3))$ then $\mathbb{P}[Y \leq (1-\delta)s(K+1/3)/(K+1)] = \mathbb{P}[X \leq s/6]$ and hence, $\mathbb{P}[X \leq s/6] \leq e^{-\delta^2 s/3}$. Now, from Theorem 3.20 we know that $s = \Omega(\varepsilon^2 \min\{\varepsilon, 1/poly(K)\} F)$ w.h.p., so $s = \omega(K \log N)$. This implies that when running the protocol for $F$ time steps, $X > K \log N$ w.h.p. Thus, if the initial value of the potential $\Phi_0$ is at most $K \log N$, we must have reached a point where $x_{\max} < x_{\min} + \log_{1+\gamma}(2K^2)$ as otherwise we would end up with a negative potential. It remains to bound $\Phi_0$.

Given that all nodes start with the same access probability $\hat{p}$, the maximum initial difference between $P_i$ and $P_j$ for any $i$ and $j$ is $N$ and therefore, $x_{\max} < x_{\min} + \log_{1+\gamma} N$. Hence, $\Phi_0 \leq K \log_{1+\gamma} N$, which implies the theorem. □

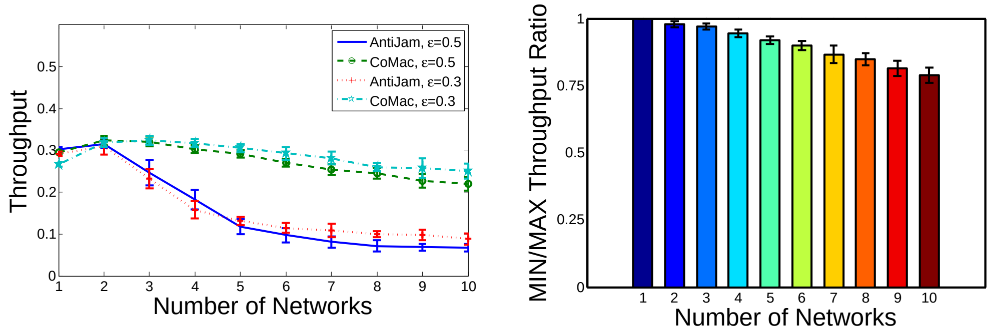


**Figure 1: *Left:* Throughput of COMAC and ANTIJAM [24] as a function of the number of co-existing networks and for two different adversaries ($\varepsilon = \{0.5, 0.3\}$). The total number of nodes for each $K = 1, \ldots, 10$ is $500$, and each co-existing network has the same size (up to an additive node due to rounding). The protocol is executed for $7000$ rounds, and the result is averaged over $10$ runs. The adversary is modeled in a simplified manner and simply jams each round with independent probability $1 - \varepsilon$. *Right:* Fairness as the min/max competitive throughput ratio for $\varepsilon = 0.3$.**

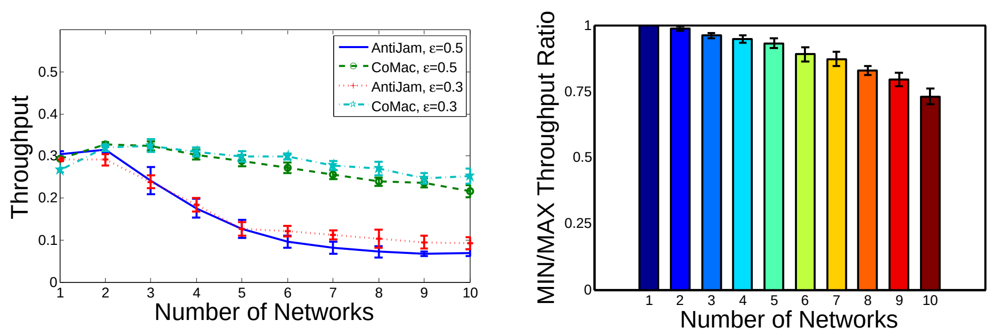


**Figure 2: *Left:* Throughput and fairness of COMAC and ANTIJAM [24] for a setting like in Figure 1 but where the size of the co-existing networks is heterogenous, i.e., the $i$-th largest network is roughly $1.5$ times the size of $(i + 1)$-largest network. *Right:* Fairness as the min/max competitive throughput ratio for $\varepsilon = 0.3$.**

Fact 3.1 ensures that the access probabilities of the nodes within a network differs by at most a $(1 + \gamma)$ factor, ensuring fairness within each network $N_i$.

# 4. SIMULATION

Although the focus of this paper is on the formal, asymptotic and worst-case performance guarantees achieved by COMAC, we also briefly report on some of our quantitative insights from a simulation study. We are interested in: $(i)$ how the competitive throughput of all the networks changes when the number of networks varies;[2] $(ii)$ the fairness of COMAC, i.e., whether the successful transmissions are evenly distributed among all the networks. Also, we compare COMAC to the state-of-the-art jamming resistent MAC protocol ANTIJAM in [24], and find that COMAC indeed better suits co-existing networks.

There is a total of 500 nodes among all the co-existing networks, and the number of networks $K$ ranges from 1 to 10. All the results are averaged over 10 runs, and the confidence intervals are provided as well. More specifically, we conduct competitive throughput and fairness experiments in two different scenarios.

[2]The competitive throughput of all the networks is defined as the fraction of non-jammed time steps that are used for successful transmissions among all $K$ networks.

*Scenario 1:* The size of individual networks are the same, namely $|V_i| \in \{\lfloor 500/K \rfloor, \lceil 500/K \rceil\}$. In Figure 1 (*left*) we study the *competitive throughput*, i.e., the fraction of non-jammed time steps that are used for successful transmissions among all $K$ networks. We observe that for a single network ($K = 1$) the competitive throughput of COMAC is relatively worse compared to ANTIJAM as $p_v$ is raised more strictly when the channel is idle. However, COMAC is always better than ANTIJAM when there is more than one network ($K > 1$) as the additional interference introduced by co-existing networks is bounded. For example, when $K = 10$, the competitive throughput of COMAC is still above 20% even when adversary can jam 70% of all time steps, while the competitive throughput of ANTIJAM is below $10\%$. Note that there is a trend towards smaller competitiveness for larger $K$, as expected from our formal worst-case analysis. Figure 1 (*right*) studies the fairness of COMAC in terms of min/max competitive throughput ratio, where the minimum and maximum competitive throughput are selected from the $K$ co-existing networks. The closer this ratio is to 1, the fairer the protocol. Obviously COMAC is fair in a sense that even when $K = 10$, the min/max competitive throughput ratio is above $0.78$.

*Scenario 2:* The size of $i$-th largest network is roughly $1.5$ times the size of $(i+1)$-th largest network. Figure 2 shows that even when the size of individual networks vary a lot, CoMAC still achieves a better competitive throughput (above 20% when $K = 10$) compared to ANTIJAM (below 10% when $K = 10$), and more importantly, CoMAC is still fair in a sense that the min/max competitive throughput ratio when $K = 10$ is still above 0.73.

# 5. CONCLUSION

Motivated by our observation that MAC algorithms optimized for a single network often yield a poor performance in scenarios with multiple co-existing networks due to too high sending probabilities, this paper presented the first protocol for provably robust, efficient and fair medium allocation among a set of co-existing networks (e.g., of a multi-nation conference or of an emergency network). Interestingly, with simple adaptions, our protocol could even be used in scenarios where the throughput is required to be distributed according to some specific proportions (i.e., not necessarily fairly) among the co-existing networks. For instance, a spectrum owner may require the co-existing networks to use only a share of the medium that corresponds to the negotiated or auctioned share. We believe that our work raises a series of interesting questions for future research. For example, we have assumed a rather naive interference model and it would be interesting to generalize our results for the SINR physical interference model.

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