A Local $O(1)$-Approximation for Dominating Sets on Bounded Genus Graphs

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Local Model

Linial 1992:

1. Input is a graph $G = (V, E)$.
2. Every vertex is an oracle which can do every computation in one unit of time.
3. Computation is synchronous and reliable.
4. In each communication round a vertex may pass arbitrary large messages to some of its neighbors.
5. Distributed complexity defined as number of communication rounds.
Complexity of Dominating Set Problem in Classical and Local Model

2. Finding dominating sets locally is hard [Kuhn et al 2010].
3. No deterministic local algorithm with approximation factor $7 - \epsilon$ for planar graphs in $O(1)$ communication rounds [Hilke et al. 2014].
4. $O(1)$-approximation in $O(1)$-communication rounds for planar graphs [Lenzen et al. 2013].
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1. **Star**: $K_{1,n}$, center of star has degree $n$.
2. **Star Contraction**: Contracting disjoint stars to their center.
3. $H$ is a depth-1 minor of a graph $G$, if it is a star contraction of a subgraph of $G$.
4. The edge density of $G$ is $\epsilon(G) = \frac{|E(G)|}{|V(G)|}$. 
Lemma 1 (Theorem 4.4.7 Mohar and Tomassen, Graph on Surfaces)

If $G$ is of genus $g$, then $G$ excludes $K_{4g+3,3}$ as minor, and thus as a depth-1 minor.

Lemma 2

If $G$ is of genus $g$, then $G$ contains at most $g$ disjoint copies of $K_{3,3}$ as minors, and in particular as depth-1 minors.
Theorem 3

There is a local $O(g)$-approximation algorithm which approximates dominating set in the class of graphs of genus at most $g$. Furthermore the number of communication rounds is in $O(g)$. 
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The algorithm consists of two phases.

1. In Phase 1, take a set $D$ in dominator set: Set of *big* vertices.
2. In Phase 2, find a dominator set for $V - N[D]$. 
Phase 1

c : the edge density of depth-1 minors of G

Choose a set $D \subseteq V(G)$ where for every $v \in D$, $N[v]$ cannot be covered by $2c$ vertices in $V(G) - \{v\}$.
Phase 2: Finding Dominators of $V - N[D]$

1. $D' := \emptyset$
2. For all $v \in V - D : \tilde{d}(v) := |N(v) - D|$.
3. $v \in V - N[D] : D' := D' \cup \{z\}$ such that $\tilde{d}(z) = \max_{u \in N[v]} \tilde{d}(u)$. 

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A Local $O(1)$-Approximation for DomSet
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Proof at a glance

1. Number of vertices in $D$ is small by a simple counting argument.

2. $D'$ is small: We use the fact that the graph is sparse and has no $K_{3,t}$. 
Size of the Set $D$ in Uniformly Sparse Graphs is Small

**Lemma 4**

$|D| \leq (2c + 1) \cdot |M|.$

Suppose $D \cap M = \emptyset$.

$E(G') \geq (2c + 1) \cdot |D|, E(G') \leq 2c \cdot (|M| + |D|) \rightarrow |D| \leq 2c \cdot |M|$
Assumption:

1. $G$ excludes $K_{3,t}$ as depth one minor.
2. Every depth-1 minor of $G$ has constant density (at most $c$).
3. $M = \{m_1, \ldots, m_{|M|}\}$ is the optimal dominating set of $G$.
4. $C_i \subseteq V - \{m_i\}$ is a set of vertices of size at most $2c$ which dominates $N[m_i]$ for $m_i \in M - D$ (otherwise empty set).
5. $C = \bigcup_{i=1}^{|M|} C_i$. 
Illustration of the C Set

Size of the Rest of Chosen Dominator Vertices is Small

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A Local $O(1)$-Approximation for DomSet
Star graph $S^G$ of $G$

$S^G$: disjoint union of $|M|$ star subgraphs $S_1, \ldots, S_{|M|}$ of $G$ with centers $m_1, \ldots, m_{|M|}$ for $m_i \in M$.

1. Start with $S^G = (V(G), \emptyset)$.
2. Add minimum number of edges from $E(G)$ to $E(S^G)$, such that $m_1, \ldots, m_{|M|}$ dominates all vertices in $S^G$. 
Star Contraction Graph $S_G^o$

2. Add every edge $\{u, v\} \in E(G)$ to $E(S_G^o)$ if $u, v \notin M$ and $u \in V(G) - D$.
3. Contract all stars $S_1, \ldots, S_{|M|}$.

Number of vertices in $S_G^o$ is $|M|$ and number of edges is at most $c|M|$.
Illustration of Star Graph of $G$
Illustration of Star Contraction Graph of $G$

Edge between two stars could be an edge from a dominator vertex to a dominated vertex.
Illustration of Star Contraction Graph of $G$
Star Contraction Graph and Different Kind of Dominations
Each $e \in E(S_G^o)$ corresponds to at most $2c(t - 1)$ dominator edges in $G$.
Total number of edges between stars is in $O(c^2 \cdot t \cdot |M|)$

1. Each gray edge corresponds to $O(c \cdot t)$ edges in $G$.
2. Total number of gray edges is $O(c \cdot |M|)$
What additional dominators exist?

1. Similarly we can count number of dominator edges between $N[D]$ and the $V - N[D]$.
2. There can be a dominator edge which connects two vertices in a star.
3. We have to count those dominator vertices which do not have any edge to other stars.
Suppose every vertex inside a star has at most $O(c) \text{ edges}$ to $M$.

2. The degree of the vertex cannot be bigger than $O(c \cdot t)$ otherwise we find $K_{3,t}$.

3. So the corresponding star is not big (at most $O(c^2 \cdot t)$).

4. Total number of stars is $|M|$ so there are at most $O(c^2 \cdot t|M|)$ such dominator edges (resp. dominator vertices).
Total Number of Edges Inside One Star is Small Otherwise There is a $K_{3,t}$. 
Lemma 5

There are at most $|M|$ vertices which have more than $2c$ edges to $M$. 
Proof of Theorem 3

Proof. If $G$ excludes $K_{3,t}$ and has edge density at most $c$:

1. There are $|M|$ stars.
2. There are $O(c^2 \cdot t|M|)$ dominator edges between stars.
3. There are $O(c|M|)$ dominator edges to center of stars.
4. There are $O(c^3|M|)$ dominator edges other than what we count.
5. Total number of dominators is in $O(c^3 \cdot t|M|)$.
6. Number of communication rounds is $O(1)$.
7. The original approximation factor is $O(c^2)$, for similitude we show this proof.

Class of graphs of genus at most $g$ has constant edge density and excludes $K_{3,4g+3}$.
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Intuitive Idea

1. Find a canonical subgraph $K_v$ for each vertex $v$: Graphs where they are minimal on some circumstances and $K_{3,3}$ is a depth-one minor of $K_v$.

2. Remove all disjoint canonical graphs.

3. Recurse for $g$ times.
Intuitive Idea

1. If we remove $g$ times a $K_{3,3}$ minors the remaining graph is planar.
2. If it is impossible to remove any depth-one minor $K_{3,3}$ at step $i \leq g$ then, the graph is locally embeddable.
3. There are at most $g$ disjoint $K_{3,3}$ minors in graph of genus $g$.
4. Run the normal algorithm on modified graph for $t := 3$.
5. Similar analysis shows that it is $24g + O(1)$-approximation.
6. Bad news: Number of communication rounds is $12g + O(1)$.
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1. Is it possible to improve the constant factor?
2. What is the biggest class of graphs which admits a constant factor approximation for MDS?
Thank you